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SUPRAFEȚE MINIMALE

Lucrare științifică

Editura Sfântul Ierarh Nicolae

ISBN 978-606-577-014-0

Referent ştiintific:

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C U P R I N S

Introducere

Ca urmare a peste doua secole de cercetare, suprafetele minimale reprezinta unul dintre cele mai bine studiate subiecte ale geometriei diferentiale, avand o multitudine de aplicatii practice in arhitectura, fizica si chimie.

Initiator este matematicianul Joseph Louis de Langrange (1736 - 1813), care a definit pentru prima oara o suprafata minimala ca fiind cea pentru care curbura medie este nula (1760). Intuitiv, dupa cum si numele sugereaza, sunt suprafetele de arie minima marginite de o curba data. Definitia lui Lagrange este insa, avantajoasa, mai intai pentru ca se poate calcula curbura in orice punct mai usor decat aria intregii suprafete, iar apoi, este independenta de curba de contur, in consecinta, suprafetele extinse la infinit pot fi, de asemenea, minimale.

Primele exemple de suprafete minimale au aparut inca din sec. XVIII, mai intai cel trivial, al suprafetei marginite de o curba plana inchisa, iar apoi, in 1776, inginerul si geometrul francez Jean Baptiste Marie Meusnier (1754 - 1793) construieste alte doua exemple: catenoidul *(sectiunea II.1.)*, singura suprafata minimala de rotatie, neplana si elicoidul *(sectiunea II.2.)* despre care s-a demonstrat ca este singura suprafata minimala riglata, neplana (Catalan, do Carmo). Urmatorul exemplu a fost publicat in 1835, fiind construit de matematicianul german Heinrich Ferdinand Scherk (1798 - 1885) *(sectiunea II.4.)*.

O contributie importanta in studiul suprafetelor minimale a fost adusa de catre fizicianul belgian Joseph Antoine Ferdinand Plateau (1801 - 1883), care, in experimentele sale, prin introducerea in solutie de apa si glicerina a unor fire sub forma unor curbe inchise obtinea suprafete minimale. A aparut, astfel, intrebarea cunoscuta sub numele „problema lui Plateau": „Exista, pentru orice curba inchisa, oricat de complicata, o suprafata de arie minima care sa o aibe drept contur?". Aceasta problema a fost rezolvata in 1931 de matematicienii Jesse Douglas (1897-1965) si Tibor Rado (1895-1965) (independent unul de altul) care au demonstrat existenta suprafetei minimale avand o curba de contur data, fara a evidentia, insa, multe proprietati geometrice ale acestor suprafete. Problema unicitatii este inca deschisa, primele conditii pentru unicitatea suprafetelor minimale in $\mathbb{R}^3$, marginite de curbe Jordan, se datoreaza lui T. Rado, J.C.C. Nitsche si A. Tromba. Incercarile experimentale ale lui Plateau au relevat totodata importanta suprafetelor minimale in teoria capilaritatii ca suprafete de energie potentiala minima. O alta observatie importanta este ca suprafetele construite in maniera experimenetelor lui Plateau au propietatea de a avea aceeasi presiune de ambele parti ale suprafetei si deci, de a fi in echilibru – intuitiv, stratul aflat in echilibru, perturbat, va reveni la starea initiala care este suprafata minimala. Tot experimental au luat nastere suprafetele cunoscute sub numele de "bulele lui Plateau". Asupra bulelor duble s-a dat conjectura ca doua parti egale de sfera avand ca frontiera comuna un disc (deci o suprafata plana) au o suprafata totala minima. Cazul celor doua parti egale ca volum a fost demonstrat in 1995 (Hass) prin reducerea problemei la un set de 200260 de integrale rezolvate cu ajutorul calculatorului. La inceputul anului 2000, Frank Morgan, Michael Hutchings, Manuel Ritoré, si Antonio Ros au demonstrat conjectura pentru bule duble oarecare. In acest caz, al celor doua parti de sfera inegale, s-a aratat ca suprafata separatoare care minimizeaza aria totala este o portiune de sfera care se intersecteaza cu cealalta suprafata sferica sub unghiul diedru de 120°. Mai mult, curbura acestei suprafete de separare este diferenta curburilor celor doua parti de sfera ce formeaza bula dubla. Problema bulelor duble ale lui Plateau a fost apoi extinsa si in spatiul 4-dimensional si pentru anumite cazuri in spatiul 5-dimensional.

Suprafetele minimale nu pot fi privite, insa, ca suprafete ce pot fi obtinute cu ajutorul ideii experimentelor lui Plateau. Fiecare portiune suficient de mica a oricarei suprafete minimale poate fi obtinuta, intr-adevar, in aceasta maniera, dar pentru suprafete mult mai largi nu mai este posibil, ceea ce conduce la ideea unui echilibru instabil al energiei potentiale. Apare, de asemenea, ideea unei noi clase de suprafete minimale ce reprezinta interesul de studiu din ultimii treizeci de ani, din punct de vedere conceptual destul de departe de intelesul initial al notiunii de „suprafata minimala", anume suprafetele minimale fara o anumita curba drept frontiera, ce pot fi extinse la infinit. Observatia naturala ca o astfel de suprafata minimala infinita, fara autointersectii, este suprafata separatoare ce imparte spatiul in doua regiuni, a fost folosita in numeroase modele fizice si in chimie, in experimente precum echilibrul polimerilor cu lanturi lungi, ale caror structuri, pentru o mai buna intelegere, sunt comparate cu suprafetele minimale triplu periodice.

Suprafetele minimale au o sfera larga de aplicabilitate si in cristalografie, de exemplu pentru cristalele zeolite, constituite dintr-un schelet de silicon, aluminiu si atomi de hidrogen, spatiul ramas fiind completat cu cristale de gheata. In timpul unei incalziri atente, apa se evapora, ramanand un schelet foarte poros folosit in schimburile de ioni, in separarea moleculelor si in cracarea uleiului. S-a constatat ca unitatile tetraedrale de structura ale sodalitelor au forma unei suprafete minimale Schwarz. Legaturi similare au fost gasite si pentru alte zeolite. O asemanare interesanta a fost gasita si in investigarea campurilor electrice ale retelelor de cristale lichide, intre suprafetele minimale si campurile de potential zero (unde punctele incarcate sunt in nodurile retelei de cristale). In aceste studii suprafetele minimale joaca rolul de modele pentru potentiale structuri spatiale, cele din viata reala fiind mai complicate, insa, decat modelele pur matematice.

Una dintre metodele de a genera noi exemple de suprafete minimale este aceea de a modifica suprafete minimale infinite existente. Aceasta incercare a fost incurajata de studiul lui Robert Osserman, care a readus in atentie o metoda a lui Karl Theodor Weierstrass (1815 - 1897), care, folosind analiza complexa, a descoperit "formulele de reprezentare" *(sectiunea V.1.)* cu ajutorul careia poate fi generata orice suprafata minimala prin alegerea unei perechi de functii complexe, dar care nu imi descrie proprietati geometrice ale suprafetei (de exemplu, autointersectiile). Cu ajutorul parametrizarilor Weierstrass, R. Osserman a reusit sa modifice suprafete minimale cunoscute facandu-le mult mai complicate, chiar daca modificarea efectuata are un efect vizibil numai pe o mica parte a suprafetei. Cu ajutorul noii teorii Osserman, s-au obtinut trinoidul (catenoidul cu trei terminatii) si binoidul (obtinut prin adaugarea a inca doua terminatii in zona cea mai ingusta a catenoidului) (Luquesio P. Jorge, William M. Meeks). De asemenea a fost descoperita una dintre cele mai interesante suprafete minimale: suprafata Costa (numita si Costa-Hoffman-Meeks), ce-a de-a treia suprafata neperiodica – alaturi de catenoid si plan *(sectiuneaV.2.6.).*

De asemenea, pot fi modificate suprafete minimale existente si prin adaugarea de „tunele". Cu ajutorul acestei metode au fost obtinute cele mai recente exemple de suprafete minimale, prin adaugarea de noi tunele verticale in tunelele orizontale ale suprafetei Schwarz. In general, pot fi complicate prin adaugarea de tunele suprafetele minimale triplu-periodice. Mai mult, s-a dovedit ca aceasta metoda nu poate fi aplicata catenoidului (rezultat demonstrat de Richard Schoen), insa, in mod surprinzator, functioneaza la catenoidul cu patru terminatii (A. Arnez, M. Steffens, C. Teitzel, si independent, J. Berglund, W. Rossman).

Lucrarea de fata reprezinta o introducere in studiul suprafetelor minimale, din a caror multitudine au fost alese exemple mai cunoscute, avand proprietati interesante, prezentate in *Cap.II.*

Interpretarea suprafetelor minimale ca fiind cele de arie minima ce pot fi construite avand drept contur curbe date, este justificata in *Cap. III* prin introducerea notiunii de *variatie normala*, o familie de suprafete $t \mapsto \mathcal{M}(t)$ reprezentand modificarile suprafetei $\mathcal{M}$ atunci cand acesteia i se impune o miscare pe o directie normala. Se arata ca suprafata $\mathcal{M}$ are curbura medie nula daca si numai daca derivata aplicatiei $t \mapsto A(t)$ (aria) se anuleaza pentru $t = 0$, adica pentru suprafata $\mathcal{M}$. Deci interpretarea cu ajutorul ariei este echivalenta cu definitia data de Lagrange.

In *Cap. IV* este folosita tehnica variabilelor complexe si, cu ajutorul notiunilor de *parametrizari izoterme* si *armonice* introduse in acest capitol, se poate construi *famila asociata* unei perechi de parametrizari izometrice minimale conjugate armonic si se pot defini suprafete minimale conjugate ale uneia date.

Ultimul capitol trateaza subiectul suprafetelor minimale cu ajutorul parametrizarilor Weierstrass care pot defini orice suprafata minimala. Atfel, sunt reluate exemple prezentate in *Cap.II* a caror definitie este obtinuta, de aceasta data, cu ajutorul reprezentarii Weierstrass si sunt prezentate noi exemple, finalizand cu *suprafata Costa* (1984), exemplu de suprafata minimala completa, elucidand presupunerea ca singurele suprafete minimale complete ce pot fi scufundate in $\mathbb{R}^3$, de gen finit, sunt planul, catenoidul si elicoidul. Mai mult, David Hoffman si W. H. Meeks au demonstrat ca exista suprafete minimale complete scufundate de orice gen k>0 cu trei terminatii dar scufundarea este imposibila pentru cele de curbura totala finita, genul 0 si trei, patru sau cinci terminatii.

Anexa 5 contine reprezentari grafice ale unor exemple de suprafete minimale, unele dintre ele fiind definite pe parcursul lucrarii, impreuna cu proprietatile mai interesante.

Multumesc domnului profesor indrumator Conf. dr. Catalin Gherghe pentru sprijinul acordat si rabdarea in intierea in studiul acestui subiect frumos, indelung cercetat, dar totusi plin de necunoscute.

CAPITOLUL I. Notiuni introductive

In acest capitol vor fi prezentate pe scurt principalele notiuni si rezultate ce vor fi folosite pe parcursul acestei lucrari, anumite completari putand fi gasite in ***Anexa 1***.

Definitie: O submultime $\mathcal{M} \subset \mathbb{R}^3$ se numeste ***suprafata diferentiabila*** (regulata, sau simplu: suprafata) daca $\forall\, p \in \mathcal{M}$ $\exists\, V$ o vecinatate deschisa a lui p in $\mathbb{R}^3$, o multime deschisa $U \subset \mathbb{R}^2$ si o aplicatie $x: U \to V \cap \mathcal{M}$ astfel incat:

1) x este diferentiabila $x(u,v)=(x^1(u,v),x^2(u,v),x^3(u,v))$
2) x este homeomorfism
3) $\forall\, q \in U$ rang $\mathcal{J}_x(q)=2$ (conditia de regularitate)

Perechea (U,x) (sau simplu, x) poarta numele de ***parametrizare***, iar $(V \cap \mathcal{M}, x^{-1})$ se numeste ***harta*** .

Fie $\mathcal{M}$ o suprafata regulata din $\mathbb{R}^3$ si un punct $p \in \mathcal{M}$.

Fie v_p un vector tangent in p la $\mathcal{M}$. Notand cu $T_p\mathcal{M}$ spatiul vectorilor tangenti in punctul p la suprafata $\mathcal{M}$, putem scrie $v_p \in T_p\mathcal{M}$. Multimea vectorilor tangenti la $\mathbb{R}^3$ in punctul $p \in \mathbb{R}^3$ se noteaza cu $T_p\mathbb{R}^3$. In general, $T_p\mathbb{R}^n = \{\ (p,v)\ ,\ v \in \mathbb{R}^n\ \}$ [1].

Pe $T_p\mathbb{R}^3$ avem produsul scalar $<(p,v),(p,w)>=<v,w>$ $\forall\ (p,v),\ (p,w) \in T_p\mathbb{R}^3$, care induce in mod natural un produs scalar in subspatiul $T_p\mathcal{M} \subset T_p\mathbb{R}^3$, $\forall\ \mathcal{M} \subset \mathbb{R}^3$ suprafata, $\forall\ p \in \mathcal{M}$.

Definitie: Fie $\mathcal{M}$ o suprafata regulata in $\mathbb{R}^3$ si un punct $p \in \mathcal{M}$. Forma biliniara $\mathrm{I}(v_p,w_p)=<v_p,w_p>$ $\forall\ v_p,w_p \in T_p\mathcal{M}$ se numeste ***prima forma fundamentala*** a suprafetei $\mathcal{M}$ in punctul p.

Observatii:

1. Prima forma fundamentala este o forma biliniara simetrica, pozitiv definita.

2. Daca v_p , $w_p \in T_p\mathcal{M}$, (U,x) parametrizare locala in jurul lui $p \in \mathcal{M}$, avem $v_p=ax_u+bx_v$, $w_p=a'x_u+b'x_v$, unde $\{x_u,x_v\}$ e baza in $T_p\mathcal{M}$ si x_u, x_v reprezinta derivatele partiale ale lui x calculate in $(u,v) \in U$ astfel incat $x(u,v)=p$.

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[1] Se considera ca $\mathbb{R}^n$ are structura naturala de spatiu vectorial real si, de asemenea $T_p\mathbb{R}^n$ are o structura de spatiu vectorial real de dimensiune n cu operatiile de adunare si inmultire cu scalari definite astfel: (p,v)+(p,w)=(p,v+w) , λ(p,v)=(p, λ v), pentru orice $p,v,w \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Atunci $<v,w>=<x_u,x_u>aa'+<x_u,x_v>ab'+<x_v,x_u>a'b+<x_v,x_v>bb'$. Pentru usurinta, se pot face notatiile: $<x_u,x_u>=E$, $<x_u,x_v>=<x_v,x_u>=F$, $<x_v ,x_v>=G$, iar E, F, G sunt numiti ***coeficientii primei forme fundamentale***.

Fiind data o parametrizare (U,x) a punctului $p\in \mathcal{M}$, putem alege un vector normal unitar in fiecare punct $q\in x(U)$ astfel: $N(q) = \dfrac{x_u \times x_v}{\|x_u \times x_v\|}(q)$. ***(normala Gauss)***

Asadar, avem o aplicatie diferentiala $N{:}x(U)\to \mathbb{R}^3$ care asociaza fiecarui punct $q\in x(U)$ un vector normal unitar $N(q)$.

Definitie: Fie $\mathcal{M} \subset \mathbb{R}^3$ o suprafata si sfera unitate $S^2=\{(x,y,z) \in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$. Aplicatia N: $\mathcal{M}\to S^2$ definita anterior se numeste ***aplicatia Gauss*** .

Aplicatia Gauss fiind diferentiabila, putem defini diferentiala lui N, dN_p: $T_p\mathcal{M}\to T_p\mathcal{M}$. [2]

Definitie: Fie $\mathcal{M}$ suprafata regulata in $\mathbb{R}^3$ si un punct $p\in \mathcal{M}$. Forma biliniara definita prin $\mathrm{II}(v_p,w_p) = - < dN_p(v_p),w_p > \ \forall\ v_p,w_p\in T_p\mathcal{M}$ se numeste ***a doua forma fundamentala*** a suprafetei $\mathcal{M}$ in punctul p.

Coeficientii celei de-a doua forme fundamentale vor fi astfel dati de expresiile

$$e = - <N_u,x_u> = <N,x_{uu}>$$
$$f = - <N_v,x_u> = <N,x_{uv}> = <N,x_{vu}> = - <N_u,x_v>$$
$$g = - <N_v,x_v> = <N,x_{vv}>$$

Fie $x(u,v)$ parametrizare in punctul $p\in \mathcal{M}$ si $c(t)=x(u(t),v(t))$ o curba parametrizata a suprafetei $\mathcal{M}$ cu $c(0)=p$. Vectorul tangent al curbei c in punctul p este $c' = x_uu' + x_vv'$ si $dN(c')=N_uu'+N_vv'$. N_u si N_v sunt in $T_p\mathcal{M}$, deci putem scrie

$$N_u= - L_{11}x_u - L_{21}x_v\ ,$$
$$N_v= - L_{12}x_u - L_{22}x_v\ . \qquad \textbf{\textit{(ecuatiile lui Weingarten)}}$$

Functiile L_{ij} definesc un endomorfism simetric L al lui $T_p\mathcal{M}$. L se numeste ***operatorul Weingarten***.
Obtinem $dN(c')=(-L_{11}u'+L_{12}v')x_u+(L_{21}u'+L_{22}v')x_v$ sau, scris matriceal

$$dN\begin{pmatrix}u'\\ v'\end{pmatrix} = -\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}\begin{pmatrix}u'\\ v'\end{pmatrix}$$

de unde se deduce ca, in baza $\{x_u,x_v\}$, dN este dat de matricea operatorului Weingarten [3].
Prin urmare, putem scrie a doua forma fundamentala cu ajutorul operatorului Weingarten:

$$\mathrm{II}(v_p,w_p) = <Lv_p ,w_p > .$$

Definitie: ***A treia forma fundamentala*** a unei suprafete $\mathcal{M} \subset \mathbb{R}^3$ in punctul p este data de aplicatia biliniara $\mathrm{III}(v_p,w_p) = < Lv_p ,Lw_p > \ \forall\ v_p,w_p\in T_p\mathcal{M}$.

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[2] dN_p masoara cum se modifica N din $N(p)$ intr-o vecinatate a lui p. In cazul curbelor, aceasta masura era data de un numar, curbura. In cazul suprafetelor, se caracterizeaza cu ajutorul unei aplicatii liniare.
[3] Completari asupra operatorului Weingarten sunt prezentate in ***Anexa 1***.

Definitie: Fie $\mathcal{M}$ suprafata regulata in $\mathbb{R}^3$, $v_p\in T_p\mathcal{M}$. Atunci functia $k_n:T_p\mathcal{M}\to\mathbb{R}$ definita prin $k_n(v_p)=\dfrac{II(v_p,v_p)}{I(v_p,v_p)}$ $\forall\, v_p\neq 0$ si $k_n(0)=0$ se numeste ***curbura normala***.

Observatii:

1. Daca $v_p\in T_p\mathcal{M}$ astfel incat $\|v_p\|=1$, atunci $k_n(v_p)=II(v_p, v_p)$.

2. Fie $c:I\to\mathcal{M}$ o curba parametrizata canonic ($\|c'(t)\|=1\ \ \forall\, t\in I$) din suprafata $\mathcal{M}\subset\mathbb{R}^3$, $p\in c$, $\{t,n,b\}$ reperul Frenet al curbei c (n vectorul normal principal in punctul p), N normala Gauss, k curbura lui c in p. Atunci curbura normala a curbei $c\subset\mathcal{M}$ in p este $k_n=k\cos(\widehat{n,N})$.[4]

Demonstratie. Aratam mai intai ca $<c'',N>=II(c',c')$. Daca $c(t)\in\mathcal{M}\ \forall\ t\in I$, vectorul viteza c' este mereu tangent la suprafata $\mathcal{M}$, de cunde $<c',N>=0$. Daca diferentiem, avem $<c'',N>+<c',dN>=0$, $<c'',N>= - <dN,c'>=II(c',c')$.

Alegand in definitie $v_p=c'$, avem $k_n=II(c',c')=<c'',N>$ (tocmai am aratat) si, folosind prima formula a lui Frenet ($T'(s)=k(s)n(s)$) avem $k_n=<c'',N>=<kn,N>=k<n,N>=k\cos(\widehat{n,N})$.

3. $k_n(u_p)=\pm k(\gamma)$, unde γ este curba canonic parametrizata obtinuta prin intersectia suprafetei $\mathcal{M}$ cu planul determinat de $v_p\in T_p\mathcal{M}$ si $N(p)$ si $\|u_p\|=1$.

Definitie: O ***curba*** $c:I\to\mathcal{M}$ se numeste ***(linie) asimptotica*** daca curbura sa normala se anuleaza in directia c': $k_n(c'(t))=0\ \ \forall\, t\in I$.

Fie $x:U\to\mathcal{M}$ parametrizare locala, $\gamma:I\to\mathcal{M}$ curba pe suprafata, $Im\,\gamma\subset h(U)$, γ parametrizata canonic. Fie $\{t,n,b\}$ reperul Frenet al curbei γ si N normala Gauss.

Definitie: Se numeste ***geodezica*** pe suprafata o curba canonic parametrizata γ pentru care ***curbura geodezica*** $k_g=<\gamma'', N\times t>$ se anuleaza in fiecare punct al curbei γ.

Observatii:

1. Produsul vectorial din definitia geodezicei se noteaza $I=N\times t$ si se numeste ***normala intrinseca*** a curbei γ.

2. γ este geodezica daca si numai daca γ'' este normal la suprafata in orice punct al curbei γ, adica γ'' este coliniar cu N.

Pentru fiecare $p\in\mathcal{M}$ exista o baza ortonormala $\{e_1,e_2\}$ a lui $T_p\mathcal{M}$ astfel incat $dN_p(e_1)=-k_1e_1$, $dN_p(e_2)= - k_2e_2$. Mai mult, k_1, k_2 ($k_1\geq k_2$) reprezinta valorea maxima si minima a curburii normale, iar $e_1,e_2\in T_p\mathcal{M}$ sunt vectorii pentru care apar aceste valori extreme. Totodata, k_1 si k_2 reprezinta valorile proprii ale operatorului Weingarten [5].

[4] Cu alte cuvinte, k_n este lungimea proiectiei vectorului kn pe normala la suprafeta in punctul p cu semnul dat de orientarea lui N.

[5] Completari asupra acestor aspecte pot fi gasite in ***Anexa 1***

Definitie: Valorile k_1 si k_2 sunt numite ***curburile principale*** in punctul $p \in \mathcal{M}$.
Vectorii unitari $e_1, e_2 \in T_p\mathcal{M}$ se numesc ***vectorii principali (directiile principale)***.

Definitie: O curba $c: I \to \mathcal{M}$ se numeste ***curba principala (linie de curbura)*** daca si numai daca vectorul viteza c' este vector principal al operatorului Weingarten; adica $Lc' = k_i c'$, unde k_i este o curbura principala.

Definitie: Fie $\mathcal{M}$ suprafata regulata in $\mathbb{R}^3$, $p \in \mathcal{M}$. Determinantul matricii aplicatiei $dN_p: T_p\mathcal{M} \to T_p\mathcal{M}$ se numeste ***curbura Gauss*** K a suprafetei $\mathcal{M}$ in punctul p, iar urma aceleiasi matrici, luata pe jumatate si cu semn schimbat poarta numele de ***curbura medie*** H a lui $\mathcal{M}$ in punctul p.

Exprimand definitia cu ajutorul curburilor principale, putem scrie:

$$K = k_1 \ k_2 , \qquad H = \frac{k_1 + k_2}{2} .$$

CAPITOLUL II. Suprafete minimale

Definitie. Exemple

Definitie: O ***suprafata*** regulata parametrizata se numeste ***minimala*** daca curbura sa medie este nula [6].
O suprafata regulata $\mathcal{M} \subset \mathbb{R}^3$ este minimala daca pentru fiecare parametrizare a sa este minimala.

O intelegere mai intuitiva a unei suprafete minimale este aceea de suprafata de arie minima printre cele avand aceeasi curba de contur. Vom arata in Cap. III, cu ajutorul notiunii de *variatie normala,* ca aceste doua definitii coincid.

II.1. Suprafata Enneper

Definita prin

$$x(u,v) = \left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - vu^2, u^2 - v^2 \right),$$

este una dintre suprafetele cu cele mai simple definitii, dar cu o trasatura interesanta: autointersectiile.

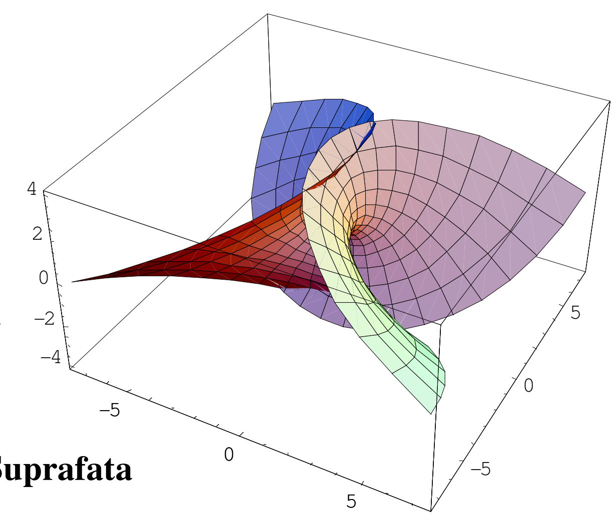


Fig. II.1.1.a. Suprafata

[6] Prima definitie a suprafetelor minimale (Lagrange, 1760).
Importanta suprafetelor minimale a fost reliefata de fizicianul belgian Joseph Antoine Ferdinand Plateau (1801-1883). In experimentele sale a format suprafete minimale prin scufundarea unor fire(sub forma de curbe in spatiu) in solutie de sapun si glicerina. Problema lui Plateau este aceea de determina suprafete minimale printr-o curba data.

Observatii:

1. Daca se modifica *(u,v)* in *(-v,u)*,atunci *(x,y,z)* se va schimba in *(-y,x,-z)*, realizandu-se astfel o rotatie cu $\pi/2$ in jurul axei *Oz* urmata de o simetrie fata de planul *xOy*. Suprafata ramane invarianta la aceasta compunere de transformari.

2. Prin transformarile $(x,y,z) \rightarrow (x,-y,z)$ si $(x,y,z) \rightarrow (-x,y,z)$ suprafata Enneper ramane invarianta. Cu alte cuvinte, ea este simetrica fata de planele *xOz* si *yOz*.

3. Rotatia cu $\pi/2$ in jurul axei *Oz* se obtine schimband *(x,y,z)* in *(y,-x,z)* – *Fig.II.2.* (iar rotatia cu unghiul $-\pi/2$ prin $(x,y,z) \rightarrow (y,x,z)$. Fiind simetrica fata de planul *xOz*, prin rotatia cu $\pi/2$ sau $-\pi/2$ suprafata va avea aceeasi reprezentare grafica).

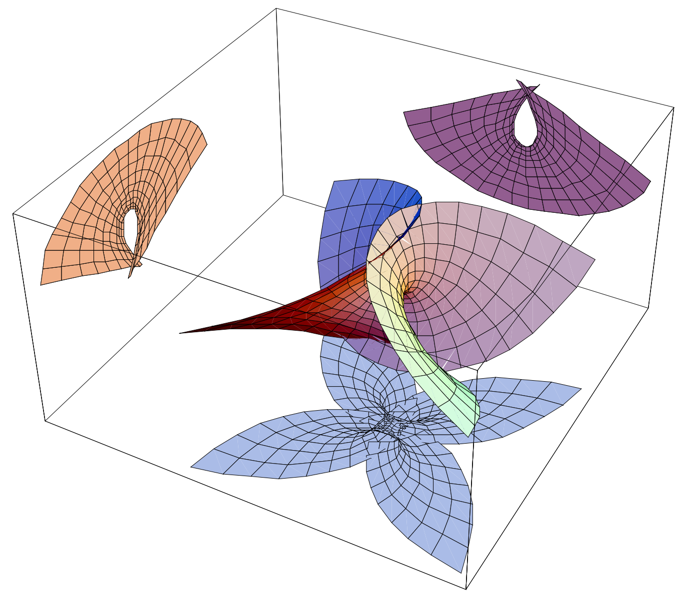


Fig. II.1.1.b. Suprafata

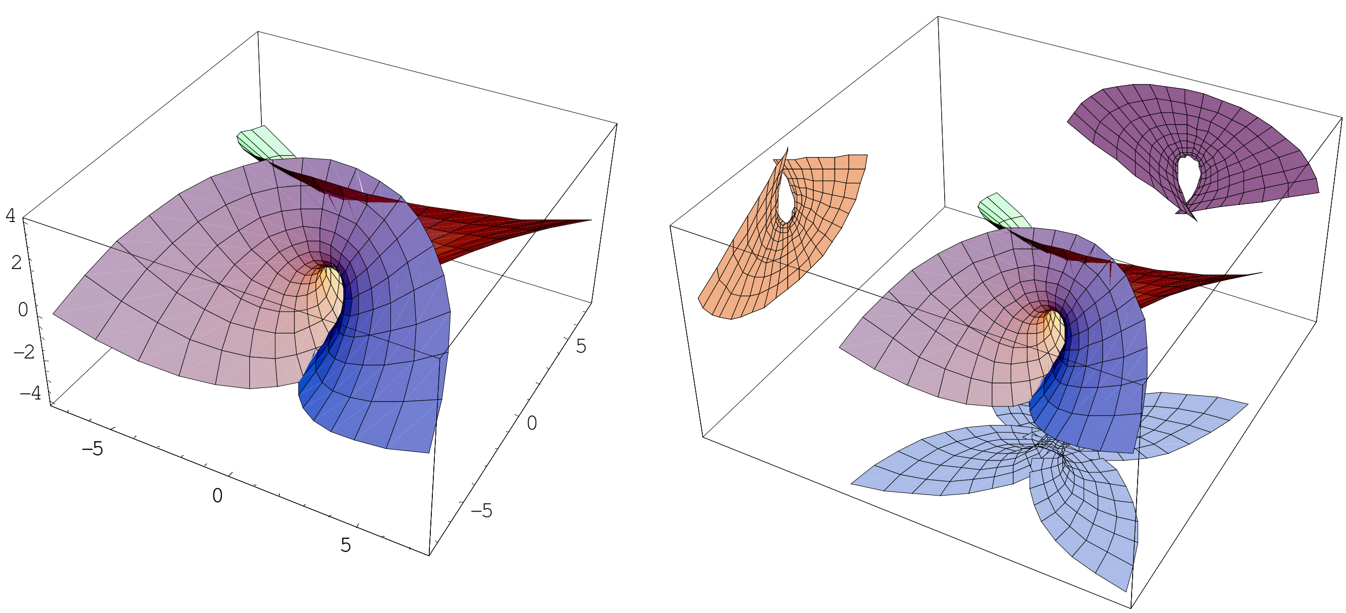


Fig. II.1.2. Rotatie cu $\pi/2$ in jurul axei *Oz*

Propozitia II.1.
Suprafata Enneper are autointersectii.
Demonstratie. Consideram $u=\rho\cos\theta$, $v=\rho\sin\theta$ si in acest caz vom scrie

$$x(\rho,\theta)=\left(\rho\cos\theta-\frac{\rho^3}{3}\cos^3\theta+\rho^3\cos\theta\sin\theta,\rho\sin\theta+\frac{\rho^3}{3}\sin^3\theta-\rho^2\sin\theta\cos^2\theta,\rho^2\cos^2\theta-\rho^2\sin^2\theta\right)=$$

$$=\left(\rho\cos\theta-\frac{\rho^3}{3}\cos\theta(\cos^2\theta-3\sin^2\theta),\rho\sin\theta+\frac{\rho^3}{3}\sin\theta(\sin^2\theta-3\cos^2\theta),\rho^2\cos(2\theta)\right).$$

Cum

$$\cos\theta(\cos^2\theta-3\sin^2\theta)=\cos\theta[(\cos^2\theta-\sin^2\theta)-2\sin^2\theta]=\cos\theta\cos(2\theta)-2(\cos\theta\sin\theta)\sin\theta=$$

$$=\frac{1}{2}(\cos(3\theta)+\cos(-\theta))+\sin(2\theta)\sin\theta=\frac{1}{2}(\cos(3\theta)+\cos\theta)+\frac{1}{2}(\cos(3\theta)-\cos\theta)=\cos(3\theta),$$

$$\sin\theta(\sin^2\theta-3\cos^2\theta)=\sin\theta[(\sin^2\theta-\cos^2\theta)-2\cos^2\theta]=-\sin\theta\cos(2\theta)-2(\sin\theta\cos\theta)\cos\theta=$$

$$=\frac{1}{2}(\sin(3\theta)-\sin(-\theta))-\sin(2\theta)\cos\theta=\frac{1}{2}(\sin(3\theta)+\sin\theta)-\frac{1}{2}(\sin(3\theta)-\sin\theta)=\sin(3\theta),$$

$$x(\rho,\theta)=\left(\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho^2\cos(2\theta)\right).$$

Daca $x(\rho_1,\theta_1)=x(\rho_2,\theta_2)$ (adica, in coordonate: $(x_1,y_1,z_1)=(x_2,y_2,z_2)$), prin calcule obtinem:

$$x_1^2+y_1^2=\left(\rho_1\cos\theta_1-\frac{\rho_1^3}{3}\cos(3\theta_1)\right)^2+\left(\rho_1\sin\theta_1+\frac{\rho_1^3}{3}\sin(3\theta_1)\right)^2=$$

$$=\rho_1^2\cos^2\theta_1+\frac{\rho_1^6}{9}\cos^2 3\theta_1-2\frac{\rho_1^4}{3}\cos\theta_1\cos(3\theta_1)+\rho_1^2\sin^2\theta_1+\frac{\rho_1^6}{9}\sin^2 3\theta_1-2\frac{\rho_1^4}{3}\sin\theta_1\sin(3\theta_1)$$

$$=\rho_1^2(\cos^2\theta_1+\sin^2\theta_1)+\frac{\rho_1^6}{9}(\cos^2 3\theta_1+\sin^2 3\theta_1)-2\frac{\rho_1^4}{3}(\cos\theta_1\cos(3\theta_1)-\sin\theta_1\sin(3\theta_1))=$$

$$=\rho_1^2+\frac{\rho_1^6}{9}-2\frac{\rho_1^4}{3}\cos(4\theta_1)=\left(\rho_1+\frac{\rho_1^3}{3}\right)^2-\frac{4}{3}(\rho_1^2\cos(2\theta_1))^2=$$

$$=\left(\rho_2+\frac{\rho_2^3}{3}\right)^2-\frac{4}{3}(\rho_2^2\cos(2\theta_2))^2=x_2^2+y_2^2.$$

Din $\rho_1^2\cos(2\theta_1)=\rho_2^2\cos(2\theta_2)$ ($z_1=z_2$) si calculele efectuate anterior rezulta ca $\left(\rho_2+\frac{\rho_2^3}{3}\right)^2=$ $\left(\rho_2+\frac{\rho_2^3}{3}\right)^2$, care implica $\rho_1=\rho_2$ si, imediat, $\cos(2\theta_1)=\cos(2\theta_2)$.

Pentru $\theta_1 = 2\pi - \theta_2$, inlocuind in $y_1 = y_2$ avem ca $\rho_1 \sin\theta_1 + \frac{\rho_1^3}{3}\sin(3\theta_1) = \rho_1 \sin(2\pi - \theta_1) + \frac{\rho_1^3}{3}\sin(3(2\pi - \theta_1))$, deci $y = -y$, de unde $y = 0$; deci punctele (ρ_1, θ_1) si (ρ_2, θ_2) apartin curbei $\sin\theta_1 + \frac{\rho_1^2}{3}\sin(3\theta_1) = 0$. Este evident ca pentru fiecare punct (ρ, θ) apartinand acestei curbe, punctul $(\rho, 2\pi - \theta)$ de asemenea apartine curbei.

Deci, intersectia suprafetei Enneper cu planul *xOz* ($y = 0$) este o curba prin care suprafata se intersecteaza cu ea insasi.

Analog (se alege $\theta_1 = \pi - \theta_2$ in egalitatea $x_1 = x_2$) se arata ca intersectia suprafetei cu planul *yOz* ($x = 0$) este o curba de autointersectie. Acestea sunt singurele autointersectii ale suprafetei Enneper.

II.2. Elicoidul

Elicoidul poate fi definit prin parametrizarea standard

$$x(u,v) = (av\cos u, av\sin u, bu)$$

sau prin reparametrizarea

$$x(u,v) = (b\,\mathrm{sh}v\cos u, b\,\mathrm{sh}v\sin u, bu),$$
$$0 < u < 2\pi,\ -\infty < v < \infty .$$

Considerand o elice data de

$$x(u,v) = (\cos u, \sin u, bu),$$

putem desena prin fiecare punct al elicei o linie paralela cu planul *xOy* care intersecteaza axa *Oz*. Suprafata generata de aceste drepte este elicoidul.

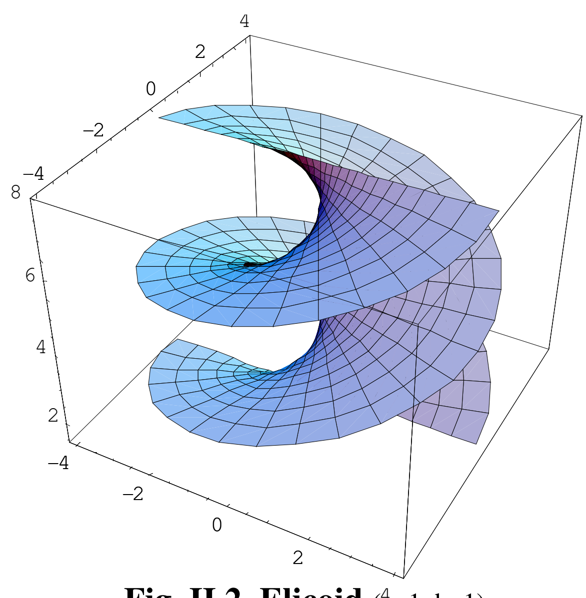


Fig. II.2. Elicoid (a=1, b=1)

Definitie: O ***familie 1-parametru de (linii) drepte*** $\{\alpha(t), w(t)\}$ este o corespondenta care atribuie fiecarui $t \in I$ un punct $\alpha(t) \in \mathbb{R}^3$ si un vector $w(t) \in \mathbb{R}^3$, $w(t) \neq 0$, atat $\alpha(t)$ cat si $w(t)$ sunt functii diferentiabile de *t*.

Pentru fiecare $t \in I$, dreapta L_t care trece prin $\alpha(t)$ si este paralela cu $w(t)$ se numeste ***dreapta familiei in t***.

Definitie: Fie familia 1-parametru de drepte $\{\alpha(t), w(t)\}$. Suprafata parametrizata $x(t,v) = \alpha(t) + vw(t)$, $t \in I$, $v \in \mathbb{R}$ se numeste ***suprafata riglata***.

Dreptele L_t se numesc ***generatoare***, iar curba $\alpha(t)$ ***(dreapta) directoare*** a suprafetei *x*.

Teorema II.2.
Elicoidul este singura suprafata minimala care este riglata, excluzand planul.

Pentru a demonstra acest fapt ne vom folosi de urmatoarele rezultate:

Teorema II.3.
Fie o suprafata regulate $\mathcal{M} \subset \mathbb{R}^3$, $c:I \to \mathcal{M}$ curba regulata, $\{t,n,b\}$ triedrul Frenet, k curbura lui c, N normala Gauss. Atunci c este curba asimptotica daca si numai daca, in fiecare punct $c(t)$, fie $k(t)=0$, fie $<n(t),N(q)>=0$.
Demonstratie. Este evidenta din Observatia 2. pentru curbura normala: $k_n=k \cos(n,N)$.

Corolar II.4.
O dreapta continuta intr-o suprafata regulata este asimptotica.
Demonstratie. Curbura unei drepte este nula.

Corolar II.5.
Generatoarele unei suprafete riglate $\mathcal{M} \subset \mathbb{R}^3$ sunt curbe asimptotice.

Lema II.6.
Curbura Gauss a unei suprafete riglate $\mathcal{M}$ din $\mathbb{R}^3$ este nepozitiva in orice punct al suprafetei.
Demonstratie. Fie $x(t,v)=\alpha(t)+vw(t)$ parametrizare a suprafetei riglate $\mathcal{M}$. Atunci $x_{v\,v}=0$, rezulta ca $g=<N, x_{v\,v}>=0$ si, folosindu-ne de *Teorema 1.1.* din *Anexa 1*, avem $K=\dfrac{-f^2}{EG-F^2}\le 0$.

Teorema II.7. (Osserman)
Fie $\mathcal{M} \subset \mathbb{R}^3$ suprafata minimala regulata, inchisa(ca submultime a lui $\mathbb{R}^3$) care nu este plana. Atunci imaginea aplicatiei Gauss N: $\mathcal{M} \to S^2$ este densa in sfera S^2.

Putem incepe acum demostratia *Teoremei II.2*.
Fie S suprafata minimala regulata, riglata. Presupunem ca suprafata nu este plana. Atunci, intr-o vecinatate V a suprafetei S curbura Gauss K este strict negativa (*Lema II.*6 si observatia ca, daca ar fi 0, cum si curbura medie este 0, ar rezulta curburile principale nule, deci S plan, contradictie cu presupunerea facuta) . Deoarece curbura medie este zero, V este acoperita de doua familii de curbe asimptotice care se intersecteaza ortogonal. Dat fiind ca generatoarele unei suprafete riglate sunt curbe asimptotice (*Corolar II.4*) si suprafata nu este plana, putem alege un punct $q\in V$ astfel incat o curba asimptotica ce trece prin punctul q, alta decat generatoarea, sa aiba torsiunea nenula in punctul q. Deoarece planul osculator al unei curbe asimptotice este planul tangent la suprafata, exista o vecinatate $W \subset V$ astfel incat generatoarele lui V sa fie normale principale la familia de curbe asimptotice torsionate. Acest fapt se intampla daca si numai daca curbele torsionate sunt elici circulare, deci V este o parte de elicoid. Cum torsiunea unei elice circulare este constanta, deducem usor ca intreaga suprafata S este o parte de elice, q.e.d. .

II.3. Catenoidul

Catenoidul este dat de

$$x(u,v)=\left(a\cos u\, ch\left(\frac{v}{a}\right), a\sin u\, ch\left(\frac{v}{a}\right), v\right),$$

$$0 < u < 2\pi,\ -\infty < v < \infty,$$

si este suprafata obtinuta prin rotatia catenei

$c(t) = \left(a\, ch\left(\frac{t}{a}\right), 0, t\right)$ in jurul axei Oz.

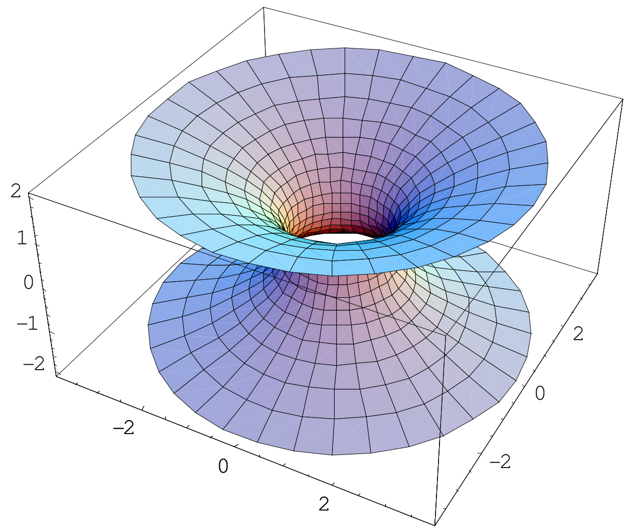


Fig. II.3. Catenoid (a=1)

Definitie: Fie Π un plan in $\mathbb{R}^3$, $d \subset \Pi$ dreapta, c o curba situata in planul Π. Suprafata $\mathcal{M}$ rezultata prin rotirea curbei c in jurul dreptei d se numeste ***suprafata de rotatie***.
Curba c este numita ***curba generatoare (profilul)*** suprafetei $\mathcal{M}$, iar dreapta d ***axa de rotatie*** a lui $\mathcal{M}$.

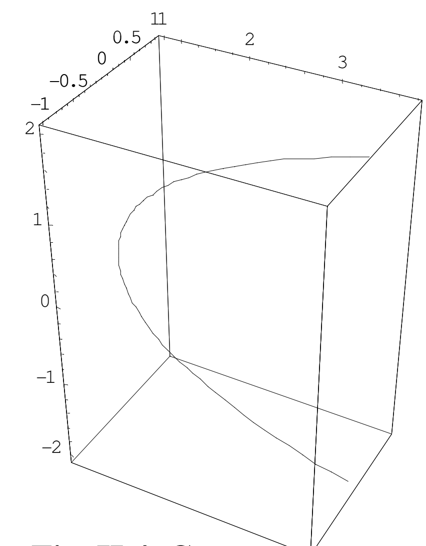


Fig. II.4. Catena (a=1)

Pentru simplitate, se alege planul Π ca fiind planul xOz, Oz axa de rotatie, curba $c : (a,b) \to \mathbb{R}^3$ parametrizata, regulata, simpla (fara autointersectii) situata in planul xOz, $c(t) = (\varphi(t), 0, \psi(t))$, $\varphi > 0$ $\forall\ t \in (a,b)$ (pentru a nu se intersecta cu axa Oz) si putem scrie parametrizarea standard a suprafetei de rotatie $x : (0,2\pi)\times(a,b) \to \mathbb{R}^3$,

$$x(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)).$$

Teorema II.8.
Catenoidul este singura suprafata de rotatie minimala, excluzand planul.
Demonstratie. Fie $\mathcal{M}$ suprafata generata prin rotirea curbei $c(t) = (\varphi(t), 0, \psi(t))$ in jurul axei Ox.
Exista trei cazuri.
Cazul 1: ψ' este functia identic nula, rezulta ca ψ este constanta, deci curba c este o dreapta paralela sau confundata (pentru ψ nula) cu axa Ox. Asadar $\mathcal{M}$ va fi un plan paralel cu planul xOy.
Cazul 2: ψ' nu este nula in niciun punct. ψ fiind inversabila, are inversa ψ^{-1}.
Definim curba $\tilde{c}(t) = c(\psi^{-1}(t)) = (f(t), 0, t)$, unde $f = \varphi \circ \psi^{-1}$ si vom avea acum $x(u,v) = (h(v)\cos u, h(v)\sin u, v)$. Deoarece $\tilde{c}$ este o reparametrizare a curbei c si cum orice reparametrizare a unei curbe are aceeasi imagine geometrica cu a curbei initiale, $\tilde{c}$ genereaza prin rotatie tot suprafata $\mathcal{M}$. Deci, este suficient sa aratam ca suprafata $x(u,v) = (h(v)\cos u, h(v)\sin u, v)$ este o parte de catenoid.

Curburile principale ale suprafetei[7] $\mathcal{M}$ sunt

$$k_1 = -\frac{h''}{(h'^2+1)^{3/2}} \quad , \qquad k_2 = \frac{1}{h\sqrt{h'^2+1}}$$

$\mathcal{M}$ fiind suprafata minimala, $H = -\frac{h''}{((h')^2+1)^{3/2}} + \frac{1}{h\sqrt{(h')^2+1}} = 0$, de unde $h''h = 1+(h')^2$. Pentru a rezolva aceasta ecuatie diferentiala, scriem $\frac{2h'h''}{1+(h')^2} = \frac{2h'}{h}$. Notand $z = 1+(h')^2$, ecuatia se scrie $\frac{z'}{z} = \frac{2h'}{h}$. Prin integrare obtinem $\ln z = 2\ln h + c = \ln(h^2) - \ln(k^2) = \ln\left(\frac{h}{k}\right)^2$, $c,k \in \mathbb{R}$. Revenind la notatiile initiale avem $\ln\ (1+(h')^2) = \ln\left(\frac{h}{k}\right)^2$, rezultand $1+(h')^2 = \left(\frac{h}{k}\right)^2$, altfel scris $\frac{h'/c}{\sqrt{(h/c)^2-1}} = \frac{1}{c}$, care prin integrare devine $\cosh^{-1}\left(\frac{h}{c}\right) = \frac{v}{c} + b$ cu solutiile $h(v) = c\cosh\left(\frac{v}{c}+b\right)$, deci $\mathcal{M}$ este o parte de catenoid.

Cazul 3: ψ' este nula in anumite puncte si nenula in altele. Dar acest caz nu poate sa apara. Sa presupunem, de exemplu, ca exista un punct v_0 pentru care $\psi'(v_0)=0$, dar $\psi'(v) > 0$ pentru $v<v_0$. Din *Cazul 2*, curba generatoare pentru $v<v_0$ este o catena cu unghiul de inclinatie dat de φ'/ψ'. Deci, $\psi'(v_0)=0$ implica faptul ca unghiul de inclinatie devine infinit in v_0, ceea ce este imposibil deoarece curba generatoare este graficul functiei *cosh*.

II.4. Parametrizari Monge. Suprafata Scherk

Definitie: Fie $h: U \to \mathbb{R}$ diferentiabila, $U \subset \mathbb{R}^2$ deschisa si suprafata data de graficul functiei h : $G_h = \mathcal{M} = \{ (x, y, h(x,y)) \mid (x,y) \in U \}$. Suprafata $\mathcal{M}$ se poate acoperi cu o singura harta a carei parametrizare este $x: U \to \mathcal{M}$, $x(u,v) = (u, v, h(u,v))$ ***(parametrizare Monge)***.

Calculele efectuate in *Anexa 2* ne ajuta sa dam urmatoarea lema.

Lema II.9.
O parametrizare Monge $x(u,v) = (u, v, h(u,v))$ este suprafata minimala daca si numai daca

$$(1+h_v^2)\, h_{uu} - 2h_u h_v h_{uv} + (1+h_u^2)\, h_{vv} = 0.$$

Un exemplu de suprafata minimala data de o parametrizare Monge este suprafata Scherk, in care parametrizarea este de forma $x(u,v) = (u, v, f(u) + g(v))$. Teorema urmatoare ne va ajuta sa definim suprafata Scherk.

[7] *Anexa 2*

Teorema II.10.
Daca o parametrizare Monge $x:U \to \mathcal{M}$, $x(u,v) = (u, v, f(u) + g(v))$ este suprafata minimala, atunci $\mathcal{M}$ este parte a unui plan sau exista o constanta nenula a astfel incat

$$f(u) = -\frac{1}{a}\log\cos(au), \qquad g(v) = \frac{1}{a}\log\cos(av).$$

Demonstratie. Fie $h(u,v) = f(u) + g(v)$. Atunci $h_u = f'(u)$, $h_v = g'(v)$, $h_{uu} = f''(u)$, $h_{uv} = 0$, $h_{vv} = g''(v)$. $\mathcal{M}$ este suprafata minimala si, folosind *Lema II.9.* obtinem $(1+(g'(u))^2)f''(u)+(1+(f'(u))^2)g''(v) = 0$, care se poate scrie sub forma $\dfrac{f''(u)}{1+(f'(u))^2} = \dfrac{-g''(v)}{1+(g'(v))^2}$,
u si v fiind variabile independente, cei doi membri ai ultimei egalitati sunt egali cu o constanta a.
Cazul 1: Daca $a = 0$ atunci f si g sunt liniare, deci $\mathcal{M}$ este parte a unui plan.
Cazul 2: Daca $a \neq 0$, prin rezolvarea ecuatiilor diferentiale $\dfrac{f''(u)}{1+(f'(u))^2} = a$, $\dfrac{-g''(v)}{1+(g'(v))^2} = a$ vom obtine f si g din concluzia teoremei.

Folosind teorema anterioara definim **suprafata** minimala **Scherk** prin

$$x(u,v) = \left(u, v, \frac{1}{a}\log\left(\frac{\cos(av)}{\cos(au)}\right)\right).$$

Pentru simplitate, vom considera in cele ce urmeaza $a = 1$.
Suprafata Scherk este bine definita pe multimea $\mathcal{P} = \{ (u,v) \mid \cos u \cos v > 0 \}$. Multimea $\mathcal{P}$ poate fi privita drept patratele negre ale unei table de sah infinita cu liniile ce delimiteaza patratele de forma

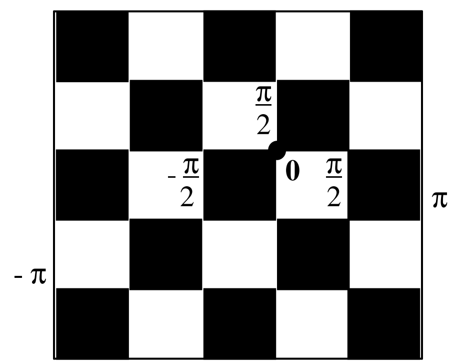
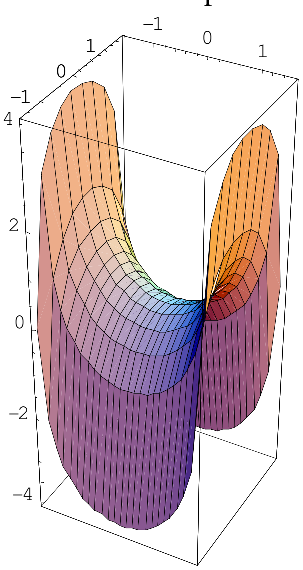


Fig. II.5.

$\{(m\pi+\frac{\pi}{2}, n\pi+\frac{\pi}{2}) \mid m,n \in \mathbb{Z}\}$. Putem considera $\mathcal{Q}(m,n)=\{(x,y) \mid m\pi - \frac{\pi}{2} < x < m\pi + \frac{\pi}{2}, n\pi - \frac{\pi}{2} < y < n\pi + \frac{\pi}{2}\}$, patrat pe care il putem colora cu negru cand $m+n$ par si cu alb cand $m+n$ impar (*Fig. II.10.*). Astfel, $\mathcal{P} = \bigcup \{ \mathcal{Q}(m,n) \mid m, n$ intregi cu $m+n$ par $\}$. Este usor de observat ca, $\forall\ u,v \in \mathbb{R}$, $\forall\ m,n \in \mathbb{Z}$ $x(u+2m\pi, v+2n\pi) = x(u,v)$. Deci, o parte a suprafetei Scherk definita pe un patrat negru $\mathcal{Q}(m,n)$ este translatie a partii din patratul $\mathcal{Q}(0,0)$. O astfel de parte apare in *Fig.II.5.*.



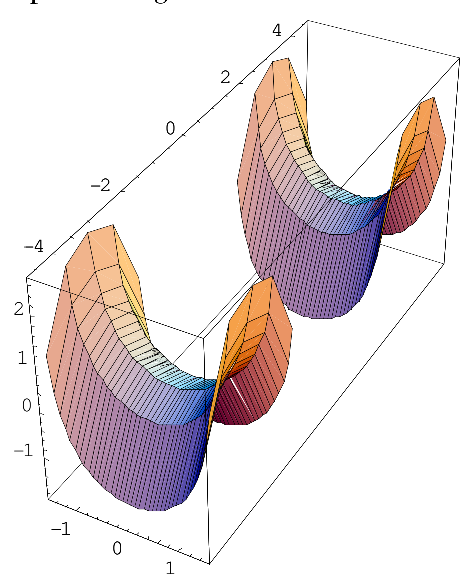


Fig. II.6. Suprafata Scherk

II.5. Suprafata Henneberg

Este definita de parametrizarea

$$x(u,v) = \left(2shu\cos v - \frac{2}{3}sh(3u)\cos(3v), 2shu\sin v + \frac{2}{3}sh(3u)\sin(3v), 2ch(2u)\cos(2v)\right).$$

Observatii:

1. Suprafata Henneberg este regulata, cu exceptia punctelor de forma $\left(0, \frac{n\pi}{2}\right)$, $n \in \mathbf{Z}$.

2. Suprafata este simetrica fata de planul xOz (ramane invarianta la transformarea $(x,y,z) \rightarrow (x,-y,z)$).

3. Deoarece $x(u,v) = x(-u,v+\pi)$, pentru orice portiune U din jumatatea dreapta a planului, $\{ (u,v) \mid v > 0 \}$ exista o portiune U' din jumatatea stanga $\{ (u,v) \mid v < 0 \}$ care au aceeasi imagine prin x. Dar normala Gauss indeplineste conditia $N(u,v) = -N(-u,v+\pi)$, deci $x(U)$ si $x(U')$ au orientari opuse.

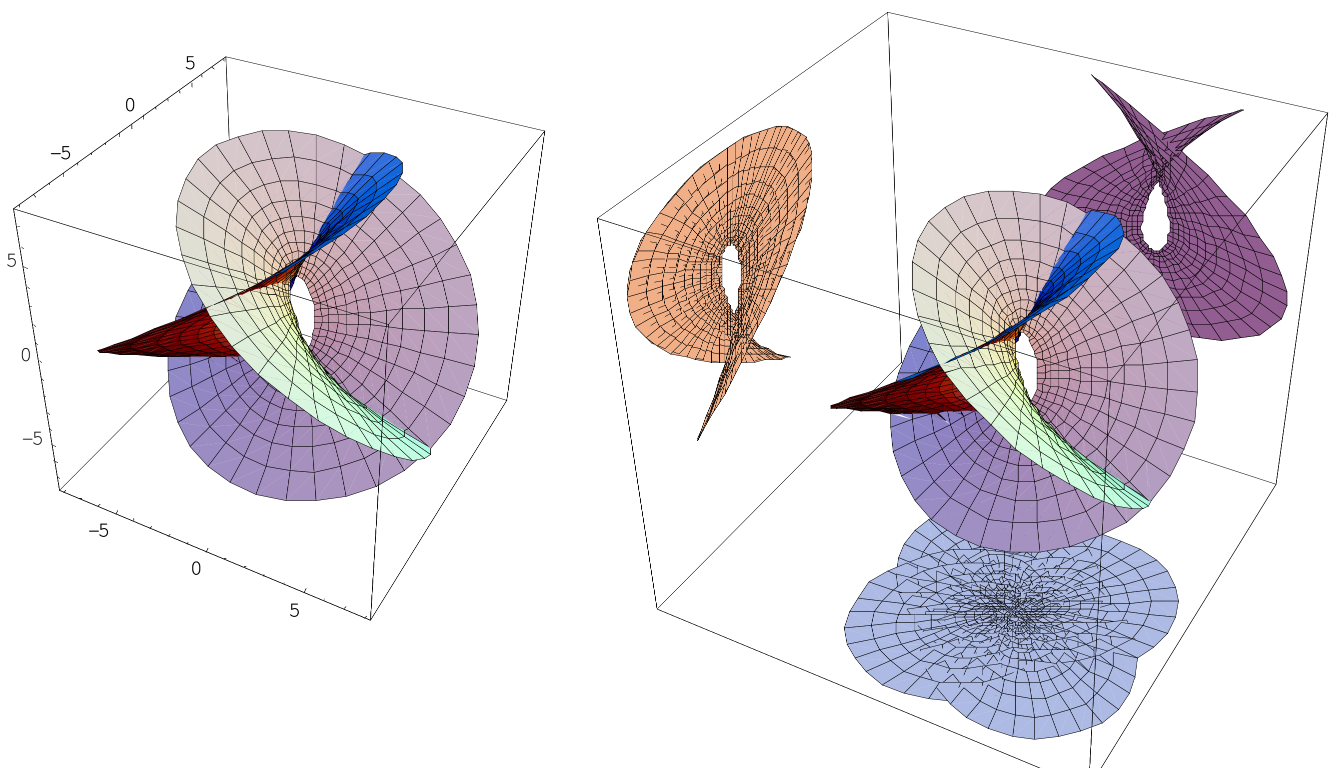


Fig. II.7. Suprafata Henneberg

II.6. Suprafata Catalan

O problema in teoria suprafatelor minimale este aceea de a determina o suprafata care contine o curba data ca geodezica, curba asimptotica sau curba principala.

Suprafata Catalan este definita prin

$$x(u,v) = a\left(u - \sin u\,ch v, 1 - \cos u\,ch v, -4\sin\frac{u}{2}\,sh\frac{v}{2}\right)$$

si contine o reparametrizare a unei cicloide *(c(t) = a(t-sin t,1-cos t),* $t\in\mathbb{R}$) ca si geodezica ($u \to x(u,0)$ este o cicloida).

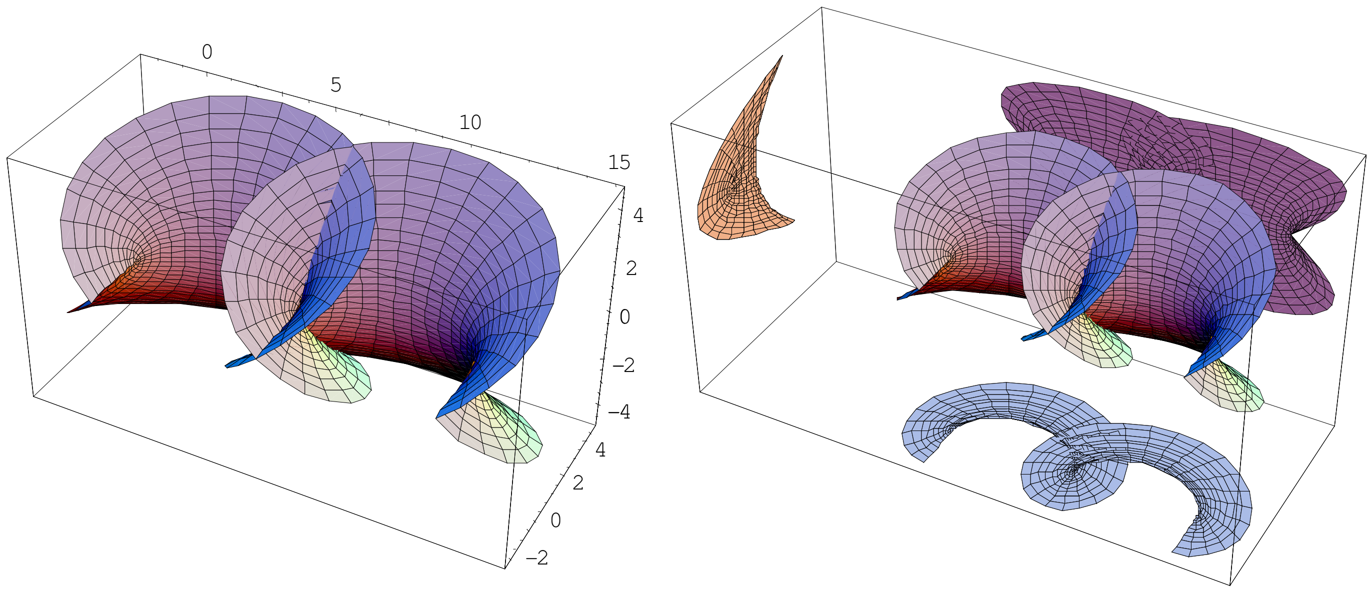


Fig. II.8. Suprafata Catalan (a=1)

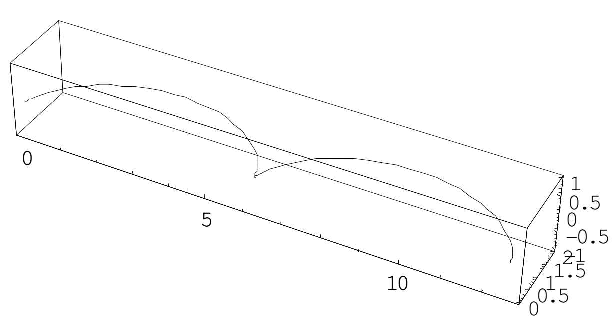


Fig. II.9. Cicloida
$u \to x(u\,,0)$, (a=1)

CAPITOLUL III. Variatia normala

In capitolul anterior suprafata minimala a fost definita ca fiind o suprafata regulata a carei curbura medie se anuleaza in orice punct. Cu ajutorul notiunii de *variatie normala* se va putea da o intrepretare mai intuitiva, explicandu-se, astfel, folosirea cuvantului „*minimal”* pentru astfel de suprafete.

Fie $U \subset \mathbb{R}^2$ multime deschisa, $\mathcal{M} \subset \mathbb{R}^3$ suprafata diferentiabila.
Fie (x,U) parametrizare, $x:U \to \mathcal{M}$.

Definitie: Fie $D \subset U$ domeniu, $h:D \to \overline{\mathbb{R}}$ aplicatie diferentabila si $\varepsilon>0$. Numim ***variatia normala*** a lui $x(D)$ determinata de h aplicatia $\varphi:D\times(-\varepsilon, \varepsilon)\to\mathbb{R}^3$, $\varphi(u,v,t)=x(u,v)+t\overline{h(u},v)N(u,v)$.
Pentru $t\in(-\varepsilon,\varepsilon)$ fixat, aplicatia $x^t:D\to\mathbb{R}^3$, $x^t(u,v)=\varphi(u,v,t)$ este suprafata.

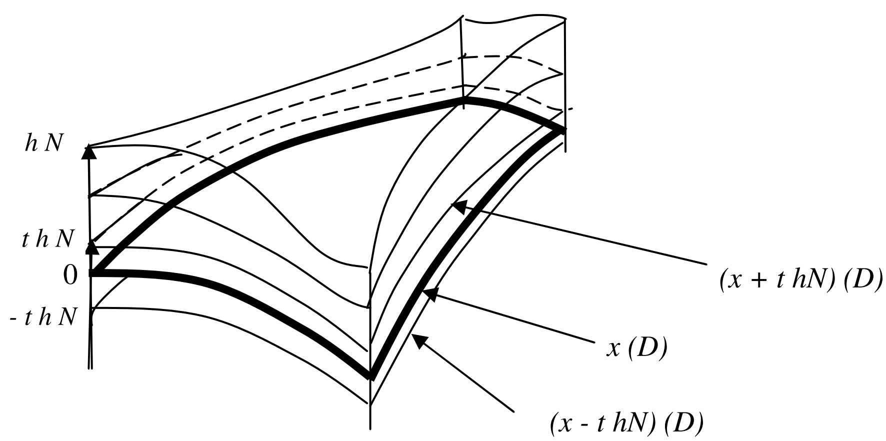


Fig. III.1. Variatia normala a lui *x(D)*

Definitie: Fie $R \subset x(U)$ domeniu. Numarul pozitiv

$$A(R) = \iint_{x^{-1}(R)} \sqrt{EG - F^2}\, dudv = \iint_{x^{-1}(R)} \sqrt{\det I}\, dudv$$

unde I reprezinta matricea coeficientilor primei forme fundamentale $E, F, G,$ se numeste ***aria*** portiunii de suprafata R.

Observatie:

Avem, in general, relatia $\|u\|^2\|v\|^2 = \|u \times v\|^2 + \langle u, v\rangle^2$, de unde $\|u \times v\|^2 = \|u\|^2\|v\|^2 - \langle u, v\rangle^2$, $\|u \times v\| = \sqrt{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}$.

Atunci $\|x_u \times x_v\| = \sqrt{\langle x_u, x_u\rangle\langle x_v, x_v\rangle - \langle x_u, x_v\rangle^2} = \sqrt{EG - F^2}$. Astfel, aria portiunii de suprafata R se mai poate scrie $A(R) = \iint_{x^{-1}(R)} \|x_u \times x_v\| dudv$.

Fie $E(t), F(t), G(t)$ coeficientii primei forme fundamentale ale suprafetei x^t: $E(t) = \langle x^t_u, x^t_u\rangle$, $F(t) = \langle x^t_u, x^t_v\rangle$, $G(t) = \langle x^t_v, x^t_v\rangle$ (observam ca $E(0)=E$, $F(0)=F$, $G(0)=G$).
Fie $D \subset U$ inchisa. Atunci aria portiunii de suprafata $x^t(D)$ este data de

$$A(t) = A(x^t(D)) = \iint_D \sqrt{E(t)G(t) - F(t)^2}\, dudv.$$

Lema III.1.
Fie $D \subset U$ domeniu, $A(t)$ aria suprafetei $x^t(D)$ (definita anterior) si H curbura medie a suprafetei $\mathcal{M}$.
Atunci $A'(0) = -2\iint_D hH\sqrt{EG - F^2}\, dudv$.

Demonstratie. Din definitie $x^t(u,v) = x(u,v) + t\, h(u,v)\, N(u,v)$ si diferentiind in raport cu u si v obtinem:

$$x^t_u = x_u + t\, h_u N + t\, h N_u$$
$$x^t_v = x_v + t\, h_v N + t\, h N_v$$

Coeficientii primei forme fundamentale ale suprafetei x^t sunt:

$$\begin{aligned}
E(t) &= \langle x^t_u, x^t_u\rangle = \langle x_u + t\, h_u N + t\, h N_u,\ x_u + t\, h_u N + t\, h N_u\rangle \\
&= \langle x_u, x_u\rangle + 2t\, h \langle x_u, N_u\rangle + t^2 (h_u)^2 + t^2 h^2 \langle N_u, N_u\rangle \\
&= E - 2t\, h\, e + t^2 [(h_u)^2 + h^2 \langle N_u, N_u\rangle] \\
&= E - 2t\, h\, e + O(t^2), \\
F(t) &= \langle x^t_u, x^t_v\rangle = \langle x_u + t\, h_u N + t\, h N_u,\ x_v + t\, h_v N + t\, h N_v\rangle \\
&= \langle x_u, x_v\rangle + 2t\, h \langle x_u, N_v\rangle + t^2 h_u h_v + t^2 h^2 \langle N_u, N_v\rangle \\
&= F - 2t\, h f + t^2 [h_u h_v + h^2 \langle N_u, N_v\rangle] \\
&= F - 2t\, h f + O(t^2), \\
G(t) &= \langle x^t_v, x^t_v\rangle = \langle x_v + t\, h_v N + t\, h N_v,\ x_v + t\, h_v N + t\, h N_v\rangle \\
&= \langle x_v, x_v\rangle + 2t\, h \langle x_v, N_v\rangle + t^2 (h_v)^2 + t^2 h^2 \langle N_v, N_v\rangle \\
&= G - 2t\, h\, g + t^2 [(h_v)^2 + h^2 \langle N_v, N_v\rangle] \\
&= G - 2t\, h\, g + O(t^2).
\end{aligned}$$

Din calculele efectuate anterior rezulta

$$\begin{aligned}
E(t)G(t) - F(t)^2 &= \{E - 2t\, h\, e + O(t^2)\}\{G - 2t\, h\, g + O(t^2)\} - \{F - 2t\, h f + O(t^2)\}^2 = \\
&= EG - F^2 - 2t\, h(Eg - 2Ff + Ge) + O(t^2).
\end{aligned}$$

Din *Anexa 1-Teorema 1.1.,* avem $H = \dfrac{eG - 2fF + gE}{2(EG - F^2)}$,de unde $2(Eg–2Ff+Ge) = 4H(EG–F^2)$. Prin inlocuire obtinem ca $E(t)G(t) – F(t)^2 = (EG – F^2)(1 – 4t\,h\,H) + O(t^2)$, de unde

$$\sqrt{E(t)G(t) - F(t)^2} = \sqrt{(EG - F^2)(1 - 4thH) + O(t^2)} = \sqrt{(EG - F^2)}\sqrt{(1 - 4thH) + O(t^2)} =$$

$$= \sqrt{(EG - F^2)}\sqrt{(1 - 4thH) + O(t^2)} =$$

$$= \sqrt{(EG - F^2)}\sqrt{(1 - 4thH + 4t^2h^2H^2) - 4t^2h^2H^2 + O(t^2)} =$$

$$= \sqrt{(EG - F^2)}(1 - 2thH) + O(t^2)\,.$$

Deci $A(t) = \iint_D \sqrt{E(t)G(t) - F(t)^2}\,dudv = \iint_D \left(\sqrt{(EG - F^2)}(1 - 2thH) + O(t^2)\right) dudv =$

$$= \iint_D \sqrt{EG - F^2}\,dudv - 2t\iint_D hH\sqrt{EG - F^2}\,dudv + O(t^2)\,.$$

Diferentiind in raport cu t si evaluand expresia in 0 obtinem

$$A'(0) = -2\iint_D hH\sqrt{EG - F^2}\,dudv \quad \text{q.e.d.}$$

Teorema III.2.
Fie $U \subset \mathbb{R}^2$ multime deschisa, $x{:}U \to \mathbb{R}^3$ o parametrizare a unei suprafete si $D \subset U$ inchisa. Atunci suprafata data de x este minimala daca si numai daca $A'(0) = 0$ pentru orice D si pentru orice variatie normala a lui $x(D)$.
Demonstratie. Implicatia directa: daca x este suprafata minimala, atunci curbura medie H este nula si $A'(0) = 0$ pentru orice D si orice h.
Implicatia inversa: presupunem prin absurd ca x nu este suprafata minimala, deci $\exists\, q \in D$ astfel incat $H(q) \neq 0$. Fie $h{:} \to \mathbb{R}$ astfel incat $h(q) = H(q)$ si h este identic nula in afara unei vecinatati a lui q. Dar, din lema anterioara rezulta ca $A'(0) < 0$, contradictie cu ipoteza. Cum q a fost ales arbitrar, rezulta ca x este suprafata minimala.,q.e.d.

CAPITOLUL IV. Coordonate izoterme

IV.1. Parametrizari izoterme. Transformari conforme

Definitie: Fie $U \subset \mathbb{R}^2$ multime deschisa. Parametrizarea $x{:}U \to \mathbb{R}^n$ se numeste ***izoterma*** daca $<x_u , x_u>=<x_v , x_v>= \lambda^2$ si $<x_u , x_v>=0$, unde $\lambda : U \to \mathbb{R}$ este functie diferentiabila numita ***functie de scalare***.

Definitie: Fie $\mathcal{M}_1$, $\mathcal{M}_2$ suprafate diferentiabile. Un difeomorfism $f : \mathcal{M}_1 \to \mathcal{M}_2$ se numeste ***transformare conforma*** daca $<df_p(v_p),df_p(w_p)> = \lambda^2(p)<v_p,w_p>$ $\forall p\in \mathcal{M}_1$, $v_p,w_p \in T_p\mathcal{M}_1$, unde λ^2 este functie diferentiabila pe $\mathcal{M}_1$. $\mathcal{M}_1$ si $\mathcal{M}_2$ se numesc ***suprafete conforme***.
Fie U vecinatate a punctului $p\in \mathcal{M}_1$. Difeomorfismul $f : U \to \mathcal{M}_2$ se numeste ***transformare local conforma*** in punctul p daca $\exists$ V vecinatate a lui $f(p)$ astfel incat $f{:}U \to V$ este transformare conforma. Daca $\forall p\in \mathcal{M}_1$ exista o transformare local conforma in p , suprafetele $\mathcal{M}_1$ si $\mathcal{M}_2$ se numesc ***suprafete local conforme***.

Observatii:
1. Local conformalitatea este o relatie de echivalenta.
2. Pentru f transformare conforma si $v_p = w_p$ se obtine $\|df_p(v_p)\| = \lambda(p)\|v_p\|$.

Din definitia transformarii conforme se deduce lema urmatoare.

Lema IV.1.
O parametrizare $x{:}U \to \mathbb{R}^3$ este izoterma daca si numai daca este conforma vazuta ca aplicatie $x{:}U \to x(U)$.

Definitie: Fie f: $U \subset \mathbb{R}^2 \to \mathbb{R}$ functie diferentiabila. ***Laplacianul*** functiei f este definit prin
$$\Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2},\ (u,v)\in U.$$

Folosind definitia data, vom calcula curbura Gauss si cea medie a unei suprafete izoterme.

Propozitia IV.2.

Curbura Gauss a unei parametrizari izoterme $x{:}U \to \mathbb{R}^3$ este $K=\frac{-\Delta \ln\lambda}{\lambda^2}$, iar curbura medie este $H=\frac{e+g}{2\lambda^2}$

Demonstratie. Din definitia parametrizarii izoterme cunoastem ca $F = <x_u,x_v> = 0$. Putem aplica *Corolarul 2.4 din Anexa 1.2.* si atunci avem formula $K = \frac{-1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right\}$,

cu $E = <x_u,x_u> = \lambda^2$, $G = <x_v,x_v> = \lambda^2$. Inlocuind, va rezulta

$$K = \frac{-1}{\lambda^2}\left\{\frac{\partial}{\partial u}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial v}\right)\right\} = \frac{-1}{\lambda^2}\left\{\frac{\partial^2}{\partial u^2}\ln\lambda+\frac{\partial^2}{\partial v^2}\ln\lambda\right\} = \frac{-\Delta\ln\lambda}{\lambda^2}.$$

Folosind *Teorema 1.1.*din *Anexa 1* si cunoscand coeficientii primei forme fundamentale ai unei parametrizari izoterme obtinem $H = \frac{eG-2fF+gE}{2(EG-F^2)} = \frac{eG+gE}{2EG} = \frac{e+g}{2\lambda^2}$.

Propozitia IV.3.

Fie $x{:}U \to \mathbb{R}^3$ parametrizarea izoterma si N normala Gauss a suprafetei. Atunci avem relatia $x_{uu}+x_{vv}=2\lambda^2 HN$.

Demonstratie. Din definitia parametrizarii izoterme cunoastem ca $<x_u,x_u> = <x_v,x_v>$ si, diferentiind in raport cu u, respectiv v, obtinem $<x_{uu},x_u>=<x_{uv},x_v>$ si $<x_{vv},x_u>= - <x_{vu},x_v>$, de unde $0 = <x_{uu},x_u> - <x_{uv},x_v> = <x_{uu},x_u> + <x_{vv},x_u> = <x_{uu}+x_{vv},x_u>$. Analog se obtine $<x_{uu}+x_{vv},x_v> = 0$. Rezulta ca $x_{uu}+x_{vv}$ este normal la suprafata $\mathcal{M}$, deci multiplu de N. Mai exact, folosind propozitia anterioara obtinem $H = \frac{e+g}{2\lambda^2} = \frac{<x_{uu}+x_{vv},N>}{2\lambda^2}$, de unde si concluzia $x_{uu}+x_{vv}=2\lambda^2 HN$.

Una dintre proprietatile interesante ale suprafetelor minimale este descrisa de urmatorul rezultat.

Lema IV.4.

Fie $x{:}U \to \mathbb{R}^3$ parametrizare a suprafetei $\mathcal{M} \subset \mathbb{R}^3$. Aplicatia Gauss este conforma daca si numai daca suprafata este sfera $\mathcal{M}$ sau suprafata minimala.

Demonstratie. Demonstram mai intai afirmatia reciproca. Fie N: $\mathcal{M} \to S^2$ aplicatia Gauss. Presupunem ca $\mathcal{M}$ este o sfera. Fara a restrange generalitatea, putem presupune ca centrul sferei $\mathcal{M}$ este in origine, astfel $x = r\,e_3$, unde r este raza sferei, e_3 vectorul normal la suprafata $\mathcal{M}$. Atunci $d\,x = r\,d\,e_3$, deci

$\mathrm{I} = r^2\,\mathrm{III}$; cum prima si a treia forma fundamentala sunt proportionale, deducem ca aplicatia Gauss este conforma.

Daca $\mathcal{M}$ este suprafata minimala, atunci avem $K<0$. Din *Corolar 1.2. (Anexa 1)* avem relatia intre cele trei forme fundamentale $III - 2H\,II + K\,I = 0$ si cum $H = 0$, obtinem $\mathrm{III} = - K\,\mathrm{I}$, ceea ce implica N transformare conforma si, in acest caz, cum $-K \geq 0$, aplicatia Gauss N are acelasi sens.

Pentru a demonstra afirmatia directa presupunem ca $\mathrm{III} = c\,\mathrm{I}$ si $H \neq 0$, $c \neq 0$ constanta. Atunci, in identitatea $III - 2H\,II + K\,I = 0$ avem $\frac{II}{I} = \frac{1}{2}\frac{(K+c)}{H}$, ceea ce arata ca fiecare punct al suprafetei $\mathcal{M}$ este punct ombilical, deci $\mathcal{M}$ este o sfera.

IV.2. Parametrizari armonice. Deformari izometrice

In aceasta sectiune ne vom folosi de elemente de analiza complexa pentru a studia suprafetele minimale.

Fie $A \subset \mathbb{C}$ deschisa, $f : A \to \mathbb{C}$ functie analitica , notam $g = \mathcal{Re}\, f$, $h = \mathcal{Im}\, f$ care verifica relatiile Cauchy-Riemann $\begin{cases} g_u = h_v \\ g_v = -h_u \end{cases}$.

In particular, pentru parametrizari ale suprafetelor, se poate da definitia urmatoare.

Definitie: Fie $U \subset \mathbb{R}^2$ multime deschisa si $x,y : U \to \mathbb{R}^n$ parametrizari. Spunem ca x si y verifica relatiile Cauchy-Riemann daca $\begin{cases} x_u = y_v \\ x_v = -y_u \end{cases}$. In acest caz x si y se numesc ***conjugate armonic***.

Observatie:
Daca x si y snt suprafete conjugate armonic, din definitie $x_u = y_v$ si $x_v = -y_u$; diferentiind, $x_{uu} = y_{uv}$ si $x_{vv} = -y_{vu}$, de unde $x_{uu} + x_{vv} = y_{uv} - y_{vu} = 0$. Analog, $y_{uu} + y_{vv} = -x_{uv} + x_{vu} = 0$.
Deci $x_{uu} + x_{vv} = y_{uu} + y_{vv} = 0$.

Definitie: Fie U multime deschisa in $\mathbb{R}^2$ si $x : U \to \mathbb{R}^n$ parametrizare. Spunem ca x este ***armonica*** daca laplacianul sau este nul: $\Delta x = x_{uu}+x_{vv} = 0$.

Lema IV.4.
Fie $U \subset \mathbb{R}^2$ multime deschisa. Daca x,y: $U \to \mathbb{R}^n$ verifica relatiile Cauchy-Riemann atunci sunt armonice
Demonstratie. Se constata usor din *Observatia* facuta anterior.

O consecinta imediata a *Propozitiei IV.3.* este:

Corolar IV.5.
Fie $U \subset \mathbb{R}^2$ multime deschisa, x:$U \to \mathbb{R}^3$ parametrizare izoterma. Atunci x este suprafata minimala daca si numai daca este armonica.

Ultimul rezultat ne ajuta sa definim suprafata minimala in $\mathbb{R}^n$ pentru orice n:

Definitie: O parametrizare $x : U \to \mathbb{R}^n$ este ***suprafata minimala izoterma*** daca este si izoterma si armonica.

In continuare, vom descrie o metoda de obtinere a unei familii 1-parametru de suprafete minimale izometrice.

Definitie: Fie x , $y : U \to \mathbb{R}^n$ parametrizari izometrice minimale conjugate armonic.
Familia 1 - parametru de suprafete $t \mapsto z(t)$ unde $z(t) : U \to \mathbb{R}^n$ este definita de $z(t) = x\cos t + y\sin t = \mathcal{Re}(e^{-it}(x+iy))$ se numeste ***familia asociata parametrizarilor*** x si y.

Lema IV.6.
Fie $U \subset \mathbb{R}^2$ multime deschisa, $x,y{:}U \to \mathbb{R}^n$ suprafete minimale izometrice conjugate armonic. Atunci familia asociata lui x si y $t \mapsto z(t)$ verifica relatiile:

$$\begin{cases} z(t)_u = x_u \cos t - x_v \sin t \\ z(t)_v = x_u \sin t + x_v \cos t \\ z(t)_{uu} = -z(t)_{vv} = x_{uu} \cos t - x_{uv} \sin t \\ z(t)_{uv} = x_{uu} \sin t - x_{uv} \cos t \end{cases}$$

iar $z(t)$ si $z(t+\pi/2)$ sunt conjugate armonic.

Demonstratie. x si y fiind cojugate armonic, verifica relatiile Cauchy-Riemann $\begin{cases} x_u = y_v \\ x_v = -y_u \end{cases}$.

Diferentiind $z(t)$ obtinem

$$z(t)_u = x_u \cos t + y_u \sin t = x_u \cos t - x_v \sin t$$
$$z(t)_v = x_v \cos t + y_v \sin t = x_v \cos t + x_u \sin t$$
$$z(t)_{uu} = x_{uu} \cos t + y_{uu} \sin t = x_{uu} \cos t - x_{uv} \sin t$$
$$z(t)_{vv} = x_{vv} \cos t + y_{vv} \sin t = -y_{uv} \cos t + x_{uv} \sin t = -x_{uu} \cos t + x_{uv} \sin t$$
$$z(t)_{uv} = x_{uv} \cos t + x_{uu} \sin t.$$

Aratam ca $z(t)$ si $z(t+\pi/2)$ sunt conjugate armonic:
$z(t)_u = \mathcal{R}e\,(e^{-it}(x_u+iy_u)) = \mathcal{R}e\,(e^{-it}(y_v - ix_v)) = \mathcal{R}e\,(e^{-i(t+\pi/2)}(x_v + iy_v)) = z(t+\pi/2)_v$.
Analog se arata $z(t)_v = -z(t+\pi/2)_u$.

Ultima lema este folositoare pentru a calcula coeficientii primei si celei de-a doua forme fundamentale ale unei familii asociate.

Teorema IV.7.
Fie $x,y{:}U \to \mathbb{R}^n$ parametrizari minimale izoterme conjugate armonic si $t \mapsto z(t)$ o familie asociata lui x si y. Atunci $z(t)$ este o suprafata izoterma minimala pentru orice t si $t \mapsto z(t)$ este o deformare izometrica.
Demonstratie. Din cea de-a treia ecuatie din *Lema IV.6.* rezulta ca $z(t)_{uu} + z(t)_{vv} = 0$, deci $z(t)$ armonica.
Folosind lema anterioara putem calcula $E(t), F(t), G(t)$ - coeficientii primei forme fundamentale ai lui $z(t)$.

$$E(t) = \langle z(t)_u, z(t)_u \rangle = \langle x_u \cos t - x_v \sin t, x_u \cos t - x_v \sin t \rangle =$$
$$= \cos^2 t \langle x_u, x_u \rangle + \sin^2 t \langle x_v, x_v \rangle - 2 \sin t \cos t \langle x_u, x_v \rangle.$$

Deoarece x este parametrizare izoterma, $\langle x_u, x_u \rangle = \langle x_v, x_v \rangle$ si $\langle x_u, x_v \rangle = 0$, deci
$E(t) = \langle x_u, x_u \rangle = E$.
$F(t) = \langle z(t)_u, z(t)_v \rangle = \langle x_u \cos t - x_v \sin t, x_v \cos t + x_u \sin t \rangle =$
$= \cos^2 t \langle x_u, x_v \rangle - \sin t \cos t \langle x_v, x_v \rangle + \sin t \cos t \langle x_u, x_u \rangle - \sin^2 t \langle x_u, x_v \rangle = 0$.
$G(t) = \langle z(t)_v, z(t)_v \rangle = \langle x_v \cos t + x_u \sin t, x_v \cos t + x_u \sin t \rangle$
$= \cos^2 t \langle x_v, x_v \rangle + \sin^2 t \langle x_u, x_u \rangle + 2 \cos t \sin t \langle x_u, x_v \rangle = \langle x_u, x_u \rangle = E$.
Deci, $E(t) = G(t)$ si $F(t) = 0$, deci $z(t)$ este izoterma si, fiind si armonica este suprafata minimala izoterma. In plus, are aceeasi prima forma fundamentala pentru orice t, deci este izometrie.

Teorema IV.8.
Fie x , $y : U \to \mathbb{R}^n$ parametrizari izometrice minimale conjugate armonic si $t \mapsto z(t)$ familia asociata. Atunci normala Gauss a suprafetei $z(t)$ in $z(t)(u,v)$ este paralel cu normala Gauss N a suprafetei x in punctul $x(u,v)$.
Demonstratie. Folosind *Lema IV.6.*

$$z(t)_u \times z(t)_v = (x_u \cos t - x_v \sin t) \times (x_v \cos t + x_u \sin t) = \cos^2 t\, (x_u \times x_v) - \sin^2 t (x_v \times x_u) = x_u \times x_v$$

de unde este evident ca normalele Gauss ale celor doua suprafete date de parametrizarile x si $z(t)$ sunt paralele.

Observatii:
1. Din teorema anterioara deducem ca spatiul tangent al suprafetei $z(t)$ in $z(t)(u,v)$ este paralel cu spatiul tangent al suprafetei x in punctul $x(u,v)$.
2. Putem identifica normala Gauss a suprafetei $z(t)$ cu normala Gauss N a suprafetei data de parametrizarea x.

Desi familia asociata unei suprafete x are aceiasi coeficienti ai primei forme fundamentale si aceeasi normala Gauss, a doua forma fundamentala este diferita. In acest sens avem urmatoarea lema.

Lema IV.9.
Fie x , $y : U \to \mathbb{R}^n$ parametrizari izometrice minimale conjugate armonic, $t \mapsto z(t)$ familia asociata. Fie $e(t)$, $f(t)$, $g(t)$ coeficientii celei de-a doua forme fundamentale ai parametrizarii $z(t)$. Atunci avem relatiile:

$$\begin{cases} e(t) = -g(t) = e\cos t - f\sin t \\ f(t) = f\cos t + e\sin t \end{cases}, \quad \begin{cases} L(t)z(t)_u = Lx_u \\ L(t)z(t)_v = Lx_v \end{cases}, \quad S(t) = (\cos t)S+(\sin t)SJ.$$

Demonstratie. Relatiile rezulta prin calcul, folosind a treia relatie din *Lema IV.6.*:

$$-g(t) = -< z(t)_{vv} , N > = < z(t)_{uu} , N > = e(t) = < x_{uu} \cos t - x_{uv} \sin t , N > =$$
$$= \cos t < x_{uu} , N > - \sin t < x_{uv} , N > = e \cos t - f \sin t,$$

unde cu e, f, g am notat coeficientii celei de-a doua forme fundamentale ai parametrizarii x.

$$f(t) = < z(t)_{v\,u} , N > = < x_{uv} \cos t + x_{uu} \sin t , N > = \cos t < x_{uv} , N > + \sin t < x_{uu} , N > =$$
$$= f \cos t + e \sin t.$$

Am obtinut, deci, primul set de ecuatii.
Din *Teorema 1.1.* din *Anexa 1* cunoastem ca

$$\begin{cases} L(t)z(t)_u = \dfrac{e(t)G(t) - f(t)F(t)}{E(t)G(t) - F(t)^2} z(t)_u + \dfrac{f(t)E(t) - e(t)F(t}{E(t)G(t) - F(t)^2} z(t)_v \\ L(t)z(t)_v = \dfrac{f(t)G(t) - g(t)F(t)}{E(t)G(t) - F(t)^2} z(t)_u + \dfrac{g(t)E(t) - f(t)F(t)}{E(t)G(t) - F(t)^2} z(t)_v \end{cases}$$

Din *Teorema IV.7.*, $t \mapsto z(t)$ este deformare izometrica, deci $E(t) = G(t) = E = \lambda^2$, $F(t) = 0$ si, prin inlocuire obtinem

$$\begin{cases} L(t)z(t)_u = \dfrac{e(t)}{\lambda^2} z(t)_u + \dfrac{f(t)}{\lambda^2} z(t)_v \\ L(t)z(t)_v = \dfrac{f(t)}{\lambda^2} z(t)_u + \dfrac{g(t)}{\lambda^2} z(t)_v \end{cases}$$

Folosind si expresiile coeficientilor celei de-a doua forme fundamentale calculate anterior si *Lema IV.6.* obtinem

$$\begin{cases} L(t)z(t)_u = \dfrac{1}{\lambda^2}\big((e\cos t - f\sin t)(x_u\cos t - x_v\sin t) + (f\cos t + e\sin t)(x_u\sin t + x_v\cos t)\big) = \dfrac{1}{\lambda^2}(ex_u + fx_v) = Lx_u \\ L(t)z(t)_v = \dfrac{1}{\lambda^2}\big((f\cos t + e\sin t)(x_u\cos t - x_v\sin t) + (f\sin t - e\cos t)(x_u\sin t + x_v\cos t)\big) = \dfrac{1}{\lambda^2}(fx_u + gx_v) = Lx_v \end{cases}$$

Din ecuatiile deduse anterior $\begin{cases} L(t)z(t)_u = \dfrac{e(t)}{\lambda^2}z(t)_u + \dfrac{f(t)}{\lambda^2}z(t)_v \\ L(t)z(t)_v = \dfrac{f(t)}{\lambda^2}z(t)_u + \dfrac{g(t)}{\lambda^2}z(t)_v \end{cases}$ si *Lema IV.6.*, rezulta ecuatiile $\begin{cases} Sx_u = (\cos t)S(t)x_u - (\sin t)S(t)x_v \\ Sx_v = (\sin t)S(t)x_u + (\cos t)S(t)x_v \end{cases}$, si rezolvand obtinem $\begin{cases} S(t)x_u = (\cos t)Sx_u + (\sin t)Sx_v \\ S(t)x_v = -(\sin t)Sx_u + (\cos t)Sx_v \end{cases}$.

Cum $x_v = J\,x_u$ putem rescrie ultimul sistem sub forma $\begin{cases} S(t)x_u = (\cos t)Sx_u + (\sin t)SJx_u \\ S(t)x_v = (\sin t)SJx_v + (\cos t)Sx_u \end{cases}$, mai scurt

S(t) = (cos t)S+(sin t)SJ.

IV.3. Derivate complexe. Complexificari

Din rezultatele sectiunii anterioare remarcam *Lema IV.6.*, care arata ca functiile armonice sunt foarte importante in studiul suprafetelor minimale.

In plus, uneori este avantajos sa trecem de la coordonatele *(u,v)* din $\mathbb{R}^2$ la coordonatele complexe z si $\bar{z}$. Formulele algebrice care leaga cele doua tipuri de coordonate sunt $\begin{cases} z = u + iv \\ \bar{z} = u - iv \end{cases}$, $\begin{cases} u = \dfrac{z+\bar{z}}{2} \\ v = \dfrac{z-\bar{z}}{2i} \end{cases}$. Putem privi z si $\bar{z}$ ca si coordonate abstracte pentru $\mathbb{R}^2 = \mathbb{C}$ si apoi sa definim u si v ca mai inainte. Asadar, putem folosi { $z, \bar{z}$ } ca sistem de coordonate pentru $\mathbb{R}^2$, in locul sistemului de coordonate standard {u,v}. Putem in contiunare folosi operatorii $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial u} - i\dfrac{\partial}{\partial v}\right)$ si $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial u} + i\dfrac{\partial}{\partial v}\right)$ si vom avea

$dz = du + i\,dv$, $d\bar{z} = du - i\,dv$, de unde $|dz|^2 = dz\,d\bar{z} = du^2 + dv^2$ si $\Delta = \dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$.

Definitie: Fie $x:U\to\mathbb{R}^n$ parametrizare. ***Derivata complexa*** a lui x este $\dfrac{\partial x}{\partial z}(z)=\dfrac{1}{2}(x_u-ix_v)(u,v)$, unde $z=u+iv$.

Putem scrie $\dfrac{\partial x}{\partial z}=\left(\phi_1(x),...,\phi_n(x)\right)=\dfrac{1}{2}\left(\dfrac{\partial x_1}{\partial u}-i\dfrac{\partial x_1}{\partial v},...,\dfrac{\partial x_n}{\partial u}-i\dfrac{\partial x_n}{\partial v}\right)$.

Lema IV.10.
Derivata complexa a unei parametrizari $x:U\to\mathbb{R}^n$ verifica

$$\sum_{k=1}^{n}\phi_k(x)^2=\frac{1}{4}(<x_u,x_u>-<x_v,x_v>-2i<x_u,x_v>)=\frac{1}{4}(E-G-2iF)\quad \text{si}$$

$$\sum_{k=1}^{n}\left|\phi_k(x)\right|^2=\frac{1}{4}\left(<x_u,x_u>+<x_v,x_v>\right)=\frac{1}{4}(E+G).$$

Demonstratie. Ambele identitati se obtin prin aplicarea definitiei si calcul direct:

$$\sum_{k=1}^{n}\phi_k(x)^2=\frac{1}{4}\sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial u}-i\frac{\partial x_k}{\partial v}\right)^2=\frac{1}{4}\sum_{k=1}^{n}\left(\left(\frac{\partial x_k}{\partial u}\right)^2-\left(\frac{\partial x_k}{\partial v}\right)^2-2i\left(\frac{\partial x_k}{\partial u}\right)\left(\frac{\partial x_k}{\partial v}\right)\right)=$$

$$=\frac{1}{4}(<x_u,x_u>-<x_v,x_v>-2i<x_u,x_v>).$$

$$\sum_{k=1}^{n}\left|\phi_k(x)\right|^2=\frac{1}{4}\sum_{k=1}^{n}\left|\frac{\partial x_k}{\partial u}-i\frac{\partial x_k}{\partial v}\right|^2=\frac{1}{4}\sum_{k=1}^{n}\left(\left|\frac{\partial x_k}{\partial u}\right|^2+\left|\frac{\partial x_k}{\partial v}\right|^2\right)=\frac{1}{4}\left(<x_u,x_u>+<x_v,x_v>\right).$$

Teorema IV.11.
Fie $x:U\to\mathbb{R}^n$ parametrizare. Atunci:

(i) x este armonica daca si numai daca derivata sa complexa $\dfrac{\partial x}{\partial z}$ este analitica;

(ii) x este izoterma daca si numai daca $\sum_{k=1}^{n}\phi_k(x)^2=0$

(iii) daca x este izoterma, atunci x este regulata daca si numai daca $\sum_{k=1}^{n}\left|\phi_k(x)\right|^2\neq 0$.

Reciproc, daca $\phi_1,...,\phi_n:\ U\to\mathbb{C}^n$ functii analitice satisfacand conditiile $\sum_{k=1}^{n}\phi_k(x)^2=0$ si $\sum_{k=1}^{n}\left|\phi_k(x)\right|^2\neq 0$, atunci exista o suprafata minimala regulata izoterma $x:U\to\mathbb{R}^n$ astfel incat ϕ = ($\phi_1,...,\phi_n$) este derivata complexa a lui x.

Demonstratie. *(i)* Observam ca relatiile Cauchy-Riemann pentru derivata complexa $\dfrac{\partial x}{\partial z}$ sunt chiar $x_{uu}+x_{vv}=0$, $x_{uv}-x_{vu}=0$, adica definitia unei parametrizari armonice.

(ii) Din lema anterioara avem ca $\sum_{k=1}^{n}\phi_k(x)^2 = \frac{1}{4}(< x_u, x_u > - < x_v, x_v > -2i < x_u, x_v >)$ = *0* $\Leftrightarrow$ $<x_u,x_u> = <x_v,x_v>$ si $<x_u,x_v> = 0$, deci x este izoterma.

(iii) Acest punct rezulta tot din *Lema IV.10.* , a doua relatie. Daca $\sum_{k=1}^{n}|\phi_k(x)|^2 = \frac{1}{4}(< x_u, x_u > + < x_v, x_v >) = 0$, atunci $<x_u,x_u> = - <x_v,x_v>$ si, impreuna cu punctul anterior *(ii)* obtinem ca $<x_u,x_u> = <x_v,x_v> = <x_u,x_v> = 0$, de unde ar rezulta ca x nu este regulata, contradictie cu ipoteza.

Pentru a demonstra reciproca, alegem $x = \mathcal{Re}\left(\int(\phi_1(z),...\phi_n(z))dz\right)$. Din ipoteza $\sum_{k=1}^{n}\phi_k(x)^2 = 0$ si punctul *(ii)* rezulta ca x este izoterma, iar din $\sum_{k=1}^{n}|\phi_k(x)|^2 \neq 0$ si *(iii)* reiese regularitatea lui x. Deoarece x este partea reala a unui n-tuplu de functii analitice, rezulta ca x este armonica. Asadar x este suprafata minimala regulata izoterma.

Un fapt important in teoria functiilor complexe este ca o pereche de functii conjugate armonice determina o functie analitica. Aceasta ne sugereaza sa gasim o functie complexa analitica pe care sa o putem asocia cu o pereche de suprafete minimale conjugate.

Definitie: Fie x , $y : U \to \mathbb{R}^n$ parametrizari izometrice minimale conjugate armonic. Numim ***complexificarea*** perechii de parametrizari x,y aplicatia $x+iy : U \to \mathbb{C}^n$.

Lema IV.12.

Complexificarea $x+iy : U \to \mathbb{C}^n$ a perechii de suprafete minimale conjugate x , $y : U \to \mathbb{R}^n$ este analitica si derivata sa complexa este $\frac{d}{dz}(x+iy) = 2\frac{\partial x}{\partial z} = x_u - ix_v$.

Demonstratie. Aplicatia $x+iy$ este analitica deoarece este un n-tuplu de functii complexe intr-o singura variabila, fiecare fiind analitica.

Derivata sa complexa va fi $\frac{d}{dz}(x+iy) = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)(x+iy) = \frac{1}{2}(x_u - ix_v + iy_v + y_v) = x_u - ix_v = 2\frac{\partial x}{\partial z}$

(am folosit faptul ca x,y conjugate armonic, deci verifica ecuatiile Cauchy-Riemann $\begin{cases} x_u = y_v \\ x_v = -y_u \end{cases}$).

IV.4. Curbe minimale

Definitie: Fie $U \subset \mathbb{C}$ submultime deschisa. O functie analitica $\psi : U \to \mathbb{C}^n$ pentru care $<\psi'(z), \psi'(z)>=0$, $z \in U$ se numeste ***curba minimala***.

Daca, in plus, $<\psi'(z), \overline{\psi'(z)}> \neq 0$, $z \in U$, numim functia analitica ψ ***curba minimala regulata***.

Observatii:

1. O curba minimala poate fi privita ca o generalizare a unei parametrizari izoterme minimale; dar, in alt sens poate fi privita ca o generalizare a unei curbe reale din $\mathbb{R}^n$.

2. O curba paremetrizata care satisface conditia din definitia curbei minimale se numeste ***izotropica***.

Lema IV.13.

O parametrizare minimala izoterma da o curba minimala si anume la complexificarea sa. Reciproc, data o curba minimala $\psi : U \to \mathbb{C}^n$, parametrizarile $x,y : U \to \mathbb{R}^n$ definite prin $x(u,v) = \mathfrak{Re}\,(\psi(u+iv))$ si $y(u,v) = \mathfrak{Im}\,(\psi(u+iv))$ sunt suprafete minimale izoterme conjugate.

Spunem ca x si y sunt ***suprafete minimale izoterme conjugate determinate de ψ.***

Demonstratie. Fie x o suprafata minimala izoterma. Prin definitie ea este izoterma si armonica. Aplicand punctele *(i)* si *(ii)* ale *Lemei IV.11.* obtinem ca $\frac{\partial x}{\partial z}$ este analitica si

$$\sum_{k=1}^{n} \phi_k(x)^2 = \frac{1}{4}(<x_u, x_u> - <x_v, x_v> - 2i <x_u, x_v>) = <x_u - ix_v, x_u - ix_v> = <\frac{d}{dz}(x+iy), \frac{d}{dz}(x+iy)> = 0$$

adica $x+iy$ (complexificarea lui x) este curba minimala.

Reciproc, din *Lema IV.11.*, x si y sunt suprafete minimale izoterme daca derivatele lor complexe sunt analitice si suma patratelor componentelor acestor derivate este 0. ψ este functie analitica, implicit partea reala si cea imaginara sunt analitice, de unde x,y armonice. ψ este curba minimala, din definitie $<\psi'(u+iv), \psi'(u+iv)>=0 \Leftrightarrow \|\psi'\|^2 = 0 \Leftrightarrow \sum_{k=1}^{n} \phi_k(\psi)^2 = 0$, deci ψ este izoterme, de unde rezulta imediat ca x,y sunt deasemenea, izoterme. In concluzie, x,y sunt suprafete minimale izoterme. In plus, se observa ca x,y verifica relatiile Cauchy-Riemann, deci x,y suprafete minimale izoterme conjugate.

Lema enuntata anterior ne arata ca a studia suprafetele minimale izoterme este echivalent cu a studia suprafetele minimale izoterme conjugate, implicit curbele minimale. Asadar, are sens notiunea de familie asociata unei curbe minimale.

Definitie: Fie $\psi : U \to \mathbb{C}^n$ curba minimala. Familia 1-parametru de suprafete $t \mapsto z(t)$ cu $z(t) : U \mapsto \mathbb{R}^n$ data de $z(t)(u,v) = \mathfrak{Re}(e^{-it}\psi(u+iv))$ se numeste ***familia asociata curbei minimale ψ.***

Observatii:
1. Familia aociata curbei minimale ψ este aceeasi familie asociata suprafetelor minimale conjugate determinate de ψ.
2. Din *Teoreme IV.7.* deducem ca familia asociata curbei minimale ψ, $t \mapsto z(t)$ este o deformare izometrica si pentru fiecare t, $z(t)$ este suprafata minimala izometrica.

Putem rescrie *Lema IV.6.* pentru familia asociata unei curbe minimale.

Corolar IV.14.
Fie $\psi : U \to \mathbb{C}^n$ curba minimala. Atunci familia asociata $t \mapsto z(t)$ satisface conditiile

$$\begin{cases} z(t)_u(u,v) = \mathcal{R}e(e^{-it}\psi'(u+iv)) \\ z(t)_v(u,v) = \mathcal{I}m(e^{-it}\psi'(u+iv)) \\ z(t)_{uu}(u,v) = -z(t)_{vv}(u,v) = \mathcal{R}e(e^{-it}\psi''(u+iv)) \\ z(t)_{uv} = \mathcal{I}m(e^{-it}\psi''(u+iv)) \end{cases}, \quad z(t)_u - iz(t)_v = e^{-it}\psi'$$

Demonstratie. Aplicand *Lema IV.12.* pentru $x(u,v)=\mathcal{R}e\,(\psi(u+iv))$ si $y(u,v)=\mathcal{I}m\,(\psi(u+iv))$ obtinem $z(t)_u - iz(t)_v = e^{-it}\psi'$, de unde rezulta primele doua relatii, iar prin diferentiere se obtin si celelalte.

Observatie:
Pentru derivata complexa a unei parametrizari $x: U \to \mathbb{R}^n$ putem rescrie rezultatul *Lemei IV.12.* sub forma:

$$\frac{\psi'}{2} = x_z = \frac{\partial x}{\partial z} = \frac{1}{2}(x_u - ix_v) = (\phi_1,\dots,\phi_n)\ ,$$
$$\frac{\overline{\psi'}}{2} = x_{\bar z} = \frac{\partial x}{\partial \bar z} = \frac{1}{2}(x_u + ix_v) = (\bar\phi_1,\dots,\bar\phi_n)\,.$$

Definitie: ***Aplicatia Gauss N a curbei minimale*** ψ este aplicatia Gauss a unuia dintre membri familiei asociate curbei ψ.

Lema IV.15.
Fie $\psi : U \to \mathbb{C}^3$ o curba minimala. Atunci

$$\frac{\psi' \times \overline{\psi'}}{4} = x_z \times x_{\bar z} = \frac{i}{2} x_u \times x_v = \frac{i}{2}(\mathcal{I}m(\phi_2\bar\phi_3), \mathcal{I}m(\phi_3\bar\phi_1), \mathcal{I}m(\phi_1\bar\phi_2)),$$
$$N = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{x_z \times x_{\bar z}}{i\|x_z\|^2} = \frac{\psi' \times \overline{\psi'}}{i\|\psi'\|^2} = \frac{2(\mathcal{I}m(\phi_2\bar\phi_3), \mathcal{I}m(\phi_3\bar\phi_1), \mathcal{I}m(\phi_1\bar\phi_2))}{\|\psi'\|^2}$$

Demonstratie. Din *Observatia* anterioara $\psi' \times \overline{\psi'} = 2x_z \times x_{\bar z}$, deci $\frac{\psi' \times \overline{\psi'}}{4} = x_z \times x_{\bar z}$;

$\psi' \times \overline{\psi'} = ((x_u - ix_v)\times(x_u + ix_v)) = (x_u \times x_u) + (x_u \times ix_v) - (ix_v \times x_u) - (ix_v \times ix_v) = 2i(x_u \times x_v)$, deci $\frac{\psi' \times \overline{\psi'}}{4} = \frac{i}{2} x_u \times x_v$.

Din *Lema 1.1.(iii), Anexa 4*, cunoastem ca $\psi'\times\overline{\psi'} = 2i(\mathcal{Im}(\phi_2\overline{\phi_3}),\mathcal{Im}(\phi_3\overline{\phi_1}),\mathcal{Im}(\phi_1\overline{\phi_2}))$, deci $\frac{\psi'\times\overline{\psi'}}{4} = \frac{i}{2}(\mathcal{Im}(\phi_2\overline{\phi_3}),\mathcal{Im}(\phi_3\overline{\phi_1}),\mathcal{Im}(\phi_1\overline{\phi_2}))$.

Din definitie, $N = \frac{x_u \times x_v}{\|x_u \times x_v\|}$.

Din $\frac{\psi'\times\overline{\psi'}}{4} = \frac{i}{2} x_u \times x_v = x_z \times x_{\bar{z}}$ rezulta $x_u \times x_v = \frac{2}{i}\, x_z \times x_{\bar{z}}$ si din *Lema 1.1.(ii), Anexa 4* avem ca

$\|x_z \times x_{\bar{z}}\| = \|x_z\|^2$, deci, prin inlocuire, $N = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{\frac{2}{i} x_z \times x_{\bar{z}}}{2\|x_z\|^2} = \frac{x_z \times x_{\bar{z}}}{i\|x_z\|^2}$.

Cum $\|\psi'\times\overline{\psi'}\| = \|\psi'\|^2$, $x_u \times x_v = \frac{1}{2i}\psi'\times\overline{\psi'}$ si $\psi'\times\overline{\psi'} = 2i(\mathcal{Im}(\phi_2\overline{\phi_3}),\mathcal{Im}(\phi_3\overline{\phi_1}),\mathcal{Im}(\phi_1\overline{\phi_2}))$ obtinem

$$N = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{\frac{1}{2i}\psi'\times\overline{\psi'}}{\frac{1}{2}\|\psi'\times\overline{\psi'}\|} = \frac{\psi'\times\overline{\psi'}}{i\|\psi'\|^2} = \frac{2\,(\mathcal{Im}(\phi_2\overline{\phi_3}),\mathcal{Im}(\phi_3\overline{\phi_1}),\mathcal{Im}(\phi_1\overline{\phi_2}))}{\|\psi'\|^2}.$$

Vom cauta in cele ce urmeaza sa dam formule pentru coeficientii formei a doua fundamentale a membrilor familiei asociate unei curbe minimale si pentru curbura Gauss.

Lema IV.16.
Fie $\psi : U \to \mathbb{C}^3$ curba minimala, *N* normala Gauss a familiei asociate $t \mapsto z(t)$. Atunci coeficientii celei de-a doua forme fundamentale a suprafetei *z(t)* sunt dati de

$$\begin{cases} e(t) = -g(t) = \mathcal{Re}(e^{-it}\psi'') \\ f(t) = -\mathcal{Im}(e^{-it}\psi'') \end{cases}.$$

Demonstratie. Din *Teorema IV.14.* avem ca $e^{-it}\psi' = z(t)_u - iz(t)_v$, de unde rezulta ca $\langle e^{-it}\psi'', N\rangle = \langle z(t)_{uu} - iz(t)_{uv}, N\rangle = \langle z(t)_{uu}, N\rangle - i\langle z(t)_{uv}, N\rangle = e(t) - i f(t)$, de unde concluzia.

Teorema IV.17.
Curbura Gauss a unei curbe minimale $\psi : U \to \mathbb{C}^n$ este

$$K = \frac{-4(\|\psi'\|^2\|\psi''\|^2 - |\langle \psi'', \overline{\psi'}\rangle|^2)}{\|\psi'\|^6}.$$

Demonstratie. Din *Propozitia IV.2.* $K = \frac{-\Delta \ln\lambda}{\lambda^2}$.

$\|\psi'\|^2 = \langle\psi', \psi'\rangle = \langle(x_u - ix_v), (x_u - ix_v)\rangle = \langle x_u, x_u\rangle + \langle x_v, x_v\rangle + 2i\langle x_u, x_v\rangle = 2\lambda^2$;

$$\Delta\log\lambda = \frac{1}{2}\Delta\log(\lambda^2) = 2\frac{\partial^2}{\partial\bar{z}\partial z}\log(\|\psi'\|^2) = 2\frac{\partial^2}{\partial\bar{z}\partial z}\log(\langle\psi, \overline{\psi'}\rangle) = 2\frac{\partial}{\partial\bar{z}}\left(\frac{\langle\psi'', \overline{\psi'}\rangle}{\|\psi'\|^2}\right) =$$

$$= 2\left(\frac{\|\psi'\|^2\|\psi''\|^2 - <\psi'',\overline{\psi'}><\psi',\overline{\psi''}>}{\|\psi'\|^4}\right) = 2\left(\frac{\|\psi'\|^2\|\psi''\|^2 - \left|<\psi'',\overline{\psi'}>\right|^2}{\|\psi'\|^4}\right).$$

Deci $K = \frac{-\Delta \ln \lambda}{\lambda^2} = -\left(\frac{2}{\|\psi'\|^2}\right)\left(\frac{2(\|\psi'\|^2\|\psi''\|^2 - \left|<\psi'',\overline{\psi'}>\right|^2)}{\|\psi'\|^4}\right) = -\frac{4(\|\psi'\|^2\|\psi''\|^2 - \left|<\psi'',\overline{\psi'}>\right|^2)}{\|\psi'\|^6}$.

IV.5. Suprafete minimale conjugate. Familii asociate

Din *Lema IV.13.* enuntata in sectiunea IV.4. deducem ca o suprafata izoterma minimala $x : U \to \mathbb{R}^n$ reprezinta partea reala a unei functii analitice $\psi : U \to \mathbb{C}^n$. Vrem insa si reciproc: avand data *x*, sa putem determina functia analitica ψ a carei parte reala sa fie *x*. In acest sens, vom demonstra urmatorul rezultat general.

Propozitia IV.18.
Fie $U \subset \mathbb{C}$ multime deschisa , $z_0 = u_0 + i\, v_0 \in U$, $h : U \to \mathbb{C}$ functie armonica. Fie $f : U \to \mathbb{C}$ o functie analitica astfel incat $\mathcal{Re}\, f(u+iv) = h(u,v)$ si $\mathcal{Im}\, f(z_0) = 0$. Atunci

$$f(z) = 2h\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2i}\right) - h(u_0, v_0) \quad .$$

Demonstratie. Putem lua $z_0 = 0$. Cum $h(u,v) = \mathcal{Re}\, f(u+iv)$, putem scrie $h(u,v) = \mathcal{Re}\frac{1}{2}(f(u+iv) + \overline{f(u+iv)})$.

Definim $g(z) = \overline{f(\overline{z})}$ (*g* este functie analitica). Atunci $h(u,v) = \mathcal{Re}\frac{1}{2}(f(u+iv) + g(u-iv))$. Pentru ca *h* este armonica, se poate dezvolta in serie in orice punct $p \in U$. Asadar, desi $z \in \mathbb{C}$, are sens $h\left(\frac{z}{2}, \frac{z}{2i}\right)$, deci putem scrie $h\left(\frac{z}{2}, \frac{z}{2i}\right) = \mathcal{Re}\frac{1}{2}(f(z) + g(0)) = \frac{1}{2}(f(z) + h(0,0))$.

Putem aplica acest rezultat in cazul parametrizarilor minimale izoterme; avem urmatorul *Corolar*.

Corolar IV.19.
Fie $U \subset \mathbb{C}$ multime deschisa, $x : U \to \mathbb{R}^n$ suprafata minimala izoterma si $\psi : U \to \mathbb{C}^n$ complexificarea parametrizarii *x* astfel incat $\mathcal{Im}\, \psi(0) = 0$. Atunci $\psi(z) = 2x\left(\frac{z}{2}, \frac{z}{2i}\right) - x(0,0)$.

Demonstratie. Prima afirmatie rezulta din lema anterioara: $x = (x_1,..., x_n)$ si ψ_j este complexificarea lui x_j cu $\mathcal{I}\boldsymbol{m}\psi_j = 0 \ \ \forall\ j = \overline{1,n}$, deci $\psi_j(z) = 2x_j\left(\frac{z}{2},\frac{z}{2i}\right) - x_j(0,0) \ \ \forall\ j = \overline{1,n}$ i.e. $\psi(z) = 2x\left(\frac{z}{2},\frac{z}{2i}\right) - x(0,0)$.

Observatia IV.20.
Parametrizarea izoterma minimala conjugata parametrizarii x cu $y(0,0) = (0,..,0)$ este
$y(u,v) = \mathcal{I}\boldsymbol{m}\left(2x\left(\frac{z}{2},\frac{z}{2i}\right) - x(0,0)\right)$.

Folosind rezultatele anterioare putem afla suprafata conjugata a unei suprafete date si, implicit, familia asociata celor doua suprafate conjugate $z(t) = x \cos t + y \sin t$. Vom relua cateva din exemplele din *Cap.II.* pentru a descrie familile asociate ale ecestora.

IV.5.1. Elicoidul si catenoidul. Familia asociata. Deformare.

Vom arata ca elicoidul si catenoidul, prin deformare, pot degenera una din cealalta, suprafetele rezultate in timpul procesului deformarii fiind suprafete minimale izometrice; de unde rezulta ca suprafetele de inceput si sfarsit in procesul deformarii, adica elicoidul si catenoidul sunt, de asemenea, izometrice.

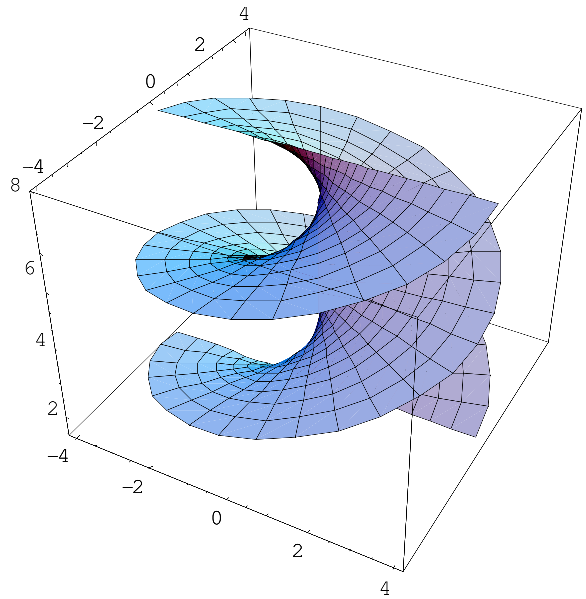
Pentru a demonstra acest lucru, consideram familia asociata elicoidului $x(u,v) = (shv\ cosu, shv\ sinu, u)$ si catenoidului $y(u,v) = (\cos u chv, a \sin u chv, v)$: $z(t)(u,v) = cost(shv\ sinu, -shv\ cosu, u) + sin\ t(chv\ cosu, chv\ sinu, v)$.

Teorema IV.21.
Familia 1-parametru de suprafete $t \to$ *elicoid-catenoid(t)* este o deformare a elicoidului in catenoid astfel incat *elicoid-catenoid(0)* este o reparametrizare a elicoidului, iar *elicoid-catenoid(*$\pi/2$*)* este un catenoid. Mai mult, pentru fiecare *t* suprafata *elicoid-catenoid(t)* este o suprafata minimala local izometrica cu *elicoid-catenoid(0)* ; in particular, elicoidul este local izometric cu catenoidul.
Demonstratie. Fie $E(t)$, $F(t)$, $G(t)$ coeficientii primei forme fundamentale ai suprafetei *elicoid-catenoid(t)*. Prin calcul [8] se obtine $E(t)(u,v)=G(t)(u,v)=ch^2v$, $F(t)(u,v)=0$. Deci $E(t)$, $F(t)$, $G(t)$ sunt functii constante de *t*, deci, pentru orice *t* suprafata *elicoid-catenoid(t)* are aceeasi coeficienti ai primei forme fundamentale, deci, pentru fiecare *t* se obtine o suprafata izometrica cu *elicoid-catenoid(0).*

Observatii:
1. Nici elicoidul nici catenoidul nu au autointersectii. Totusi, suprafetele intermediare care apar in timpul deformarii au autointersectii.
2. Prin deformare curbele asimptotice ale elicoidului se transforma in curbe principale ale catenoidului.

[8] ***Anexa 3.1***.

Figurile urmatoare ilustreaza deformarea elicoidului in catenoid:



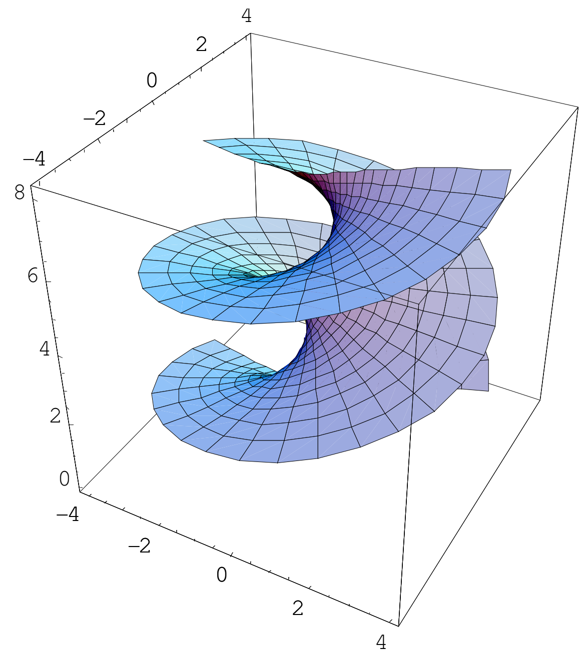


Fig. IV.1.a. Elicoidul
elicoid-catenoid(0)

elic

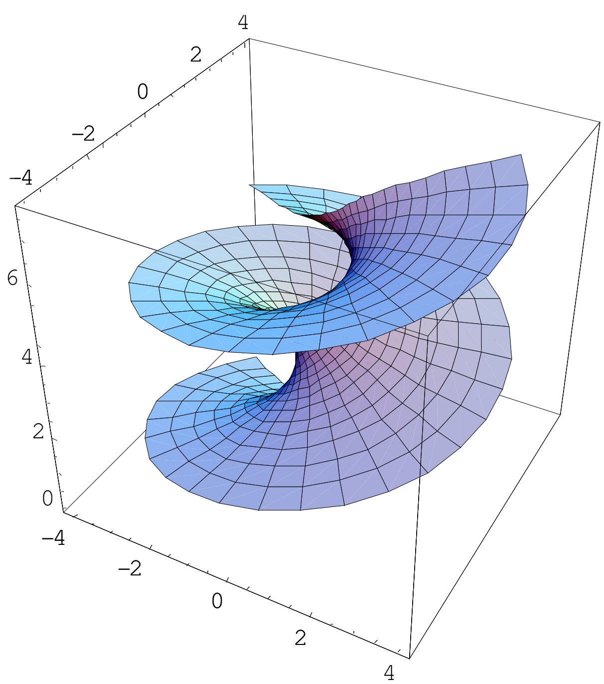
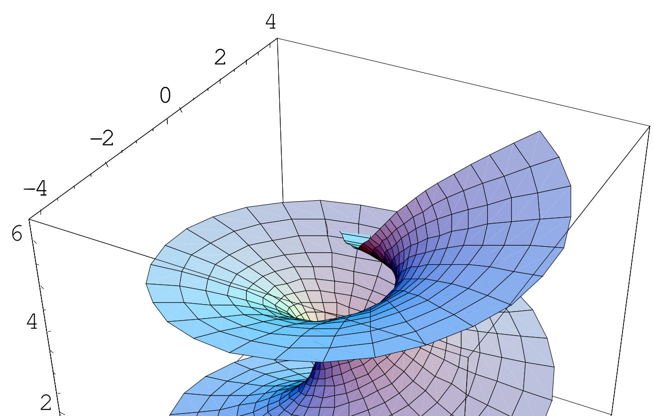


Fig. IV.1.c.
elicoid-catenoid($\pi / 5$)



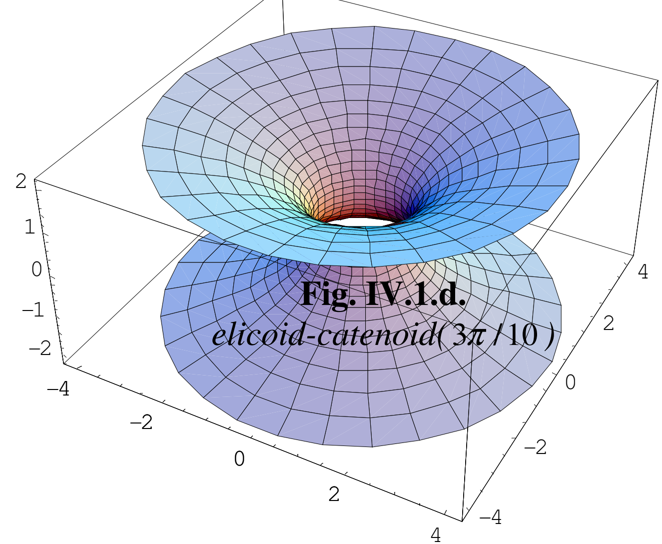


Fig. IV.1.d.
elicoid-catenoid($3\pi / 10$)

IV.5.2. Familia asociata suprafetei Catalan

Pentru a gasi familia asociata unei suprafetei Catalan *x(u,v)*=$a\left(u-\sin u ch v, 1-\cos u ch v, -4\sin\frac{u}{2} sh\frac{v}{2}\right)$ vom afla mai intai suprafata conjugata *y(u,v)*, folosind *Observatia IV.20.* si, astfel, putem defini familia asociata *z(t)=cos t x(u,v)+sin t y(u,v).*

Asadar, conform *Observatiei IV.20* suprafata conjugata suprafetei Catalan este

$$y(u,v) = \mathfrak{Im}\left(2x\left(\frac{z}{2},\frac{z}{2i}\right)-x(0,0)\right) = \mathfrak{Im}\left(2a\left(\frac{z}{2}-\sin\frac{z}{2}ch\frac{z}{2i}, 1-\cos\frac{z}{2}ch\frac{z}{2i}, -4\sin\frac{z}{4}sh\frac{z}{4i}\right)\right) =$$

$$= \mathfrak{Im}\left(2a\left(\frac{u+iv}{2}-\sin\frac{z}{2}ch\left(-\frac{iz}{2}\right), 1-\cos\frac{z}{2}ch\left(-\frac{iz}{2}\right), -4\sin\frac{z}{4}sh\left(-\frac{iz}{4}\right)\right)\right).$$

Cum $ch\left(-\frac{iz}{2}\right)=\cos\frac{z}{2}$ si $sh\left(-\frac{iz}{4}\right)=i\sin\frac{z}{4}$, rezulta

$$y(u,v) = \mathfrak{Im}\left(2a\left(\frac{u+iv}{2}-\sin\frac{z}{2}\cos\frac{z}{2}, 1-\cos\frac{z}{2}\cos\frac{z}{2}, -4\sin\frac{z}{4}i\sin\frac{z}{4}\right)\right).$$

Dar $\sin\frac{z}{2}\cos\frac{z}{2}=\frac{\sin z}{2}$, $\left(\cos\frac{z}{2}\right)^2=\frac{1+\cos z}{2}$, $\left(\sin\frac{z}{4}\right)^2=\frac{1-\cos\frac{z}{2}}{2}$, deci

$$y(u,v) = \mathfrak{Im}\left(2a\left(\frac{u+iv}{2}-\frac{\sin z}{2}, 1-\frac{1+\cos z}{2}, -2i\left(1-\cos\frac{z}{2}\right)\right)\right)=$$

$$= a\left(v-2\mathfrak{Im}\left(\frac{\sin z}{2}\right), -2\mathfrak{Im}\left(\frac{\cos z}{2}\right), 4-4\,\mathfrak{Im}\left(i\cos\frac{z}{2}\right)\right)$$

Cum $\quad \sin z = i\, sh\, v \cos u + ch\, v \sin u$ si
$\cos z = ch\, v \cos u - i\, sh\, v \sin u$ [9] deducem

$$y(u,v)=a\left(v-\cos u\, sh\, v, \sin u\, sh\, v, 4 - 4\cos\frac{u}{2}\, ch\frac{v}{2} \right).$$

Familia asociata celor doua suprafete conjugate este

$$z(t)=a \cos t\left(u-\sin u\, ch v, 1-\cos u\, ch v, -4\sin\frac{u}{2}\, sh\frac{v}{2}\right)+a$$

$$\sin t\left(v-\cos u\, sh v, \sin u\, sh v, 4-4\cos\frac{u}{2}\, ch\frac{v}{2}\right)$$

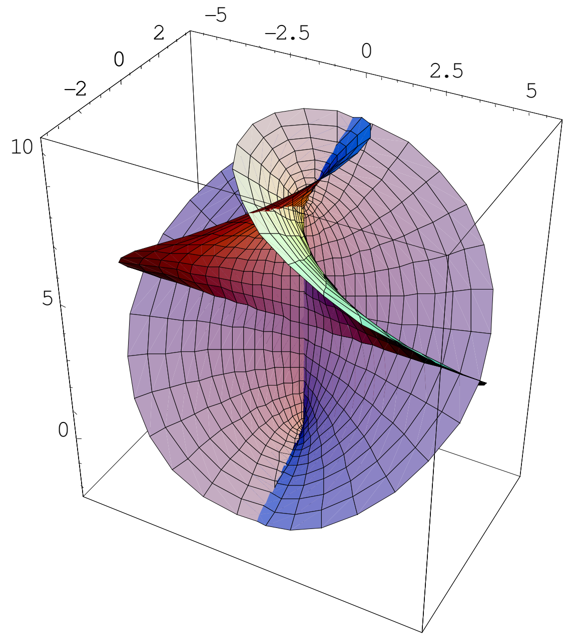
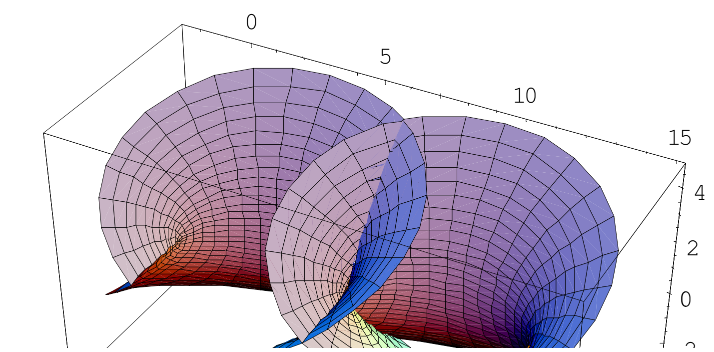


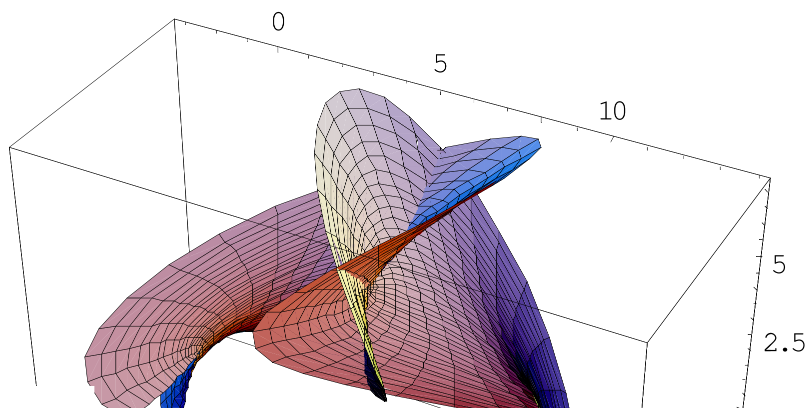
Fig. IV.2. Conjugata suprafetei Catalan

Observatie:

z(0) reprezinta suprafata Catalan, iar $z\left(\frac{\pi}{2}\right)$ este suprafata sa conjugata.

Cateva suprafete din procesul deformarii suprafetei Catalan:





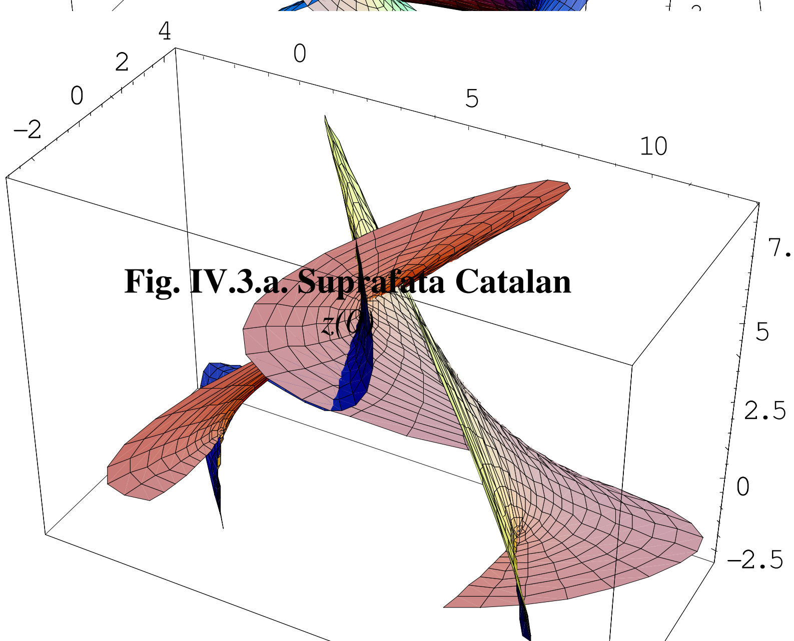


Fig. IV.3.a. Suprafata Catalan
$z(0)$

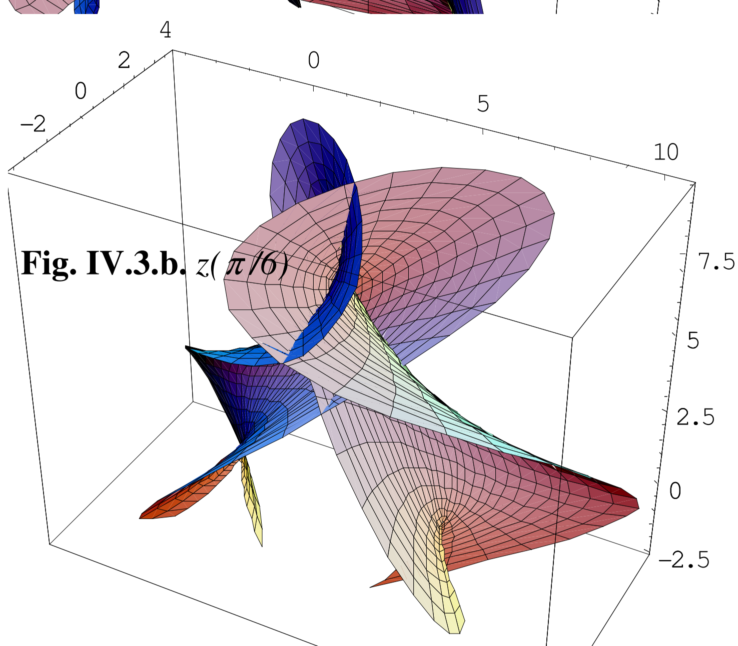


Fig. IV.3.b. $z(\pi/6)$

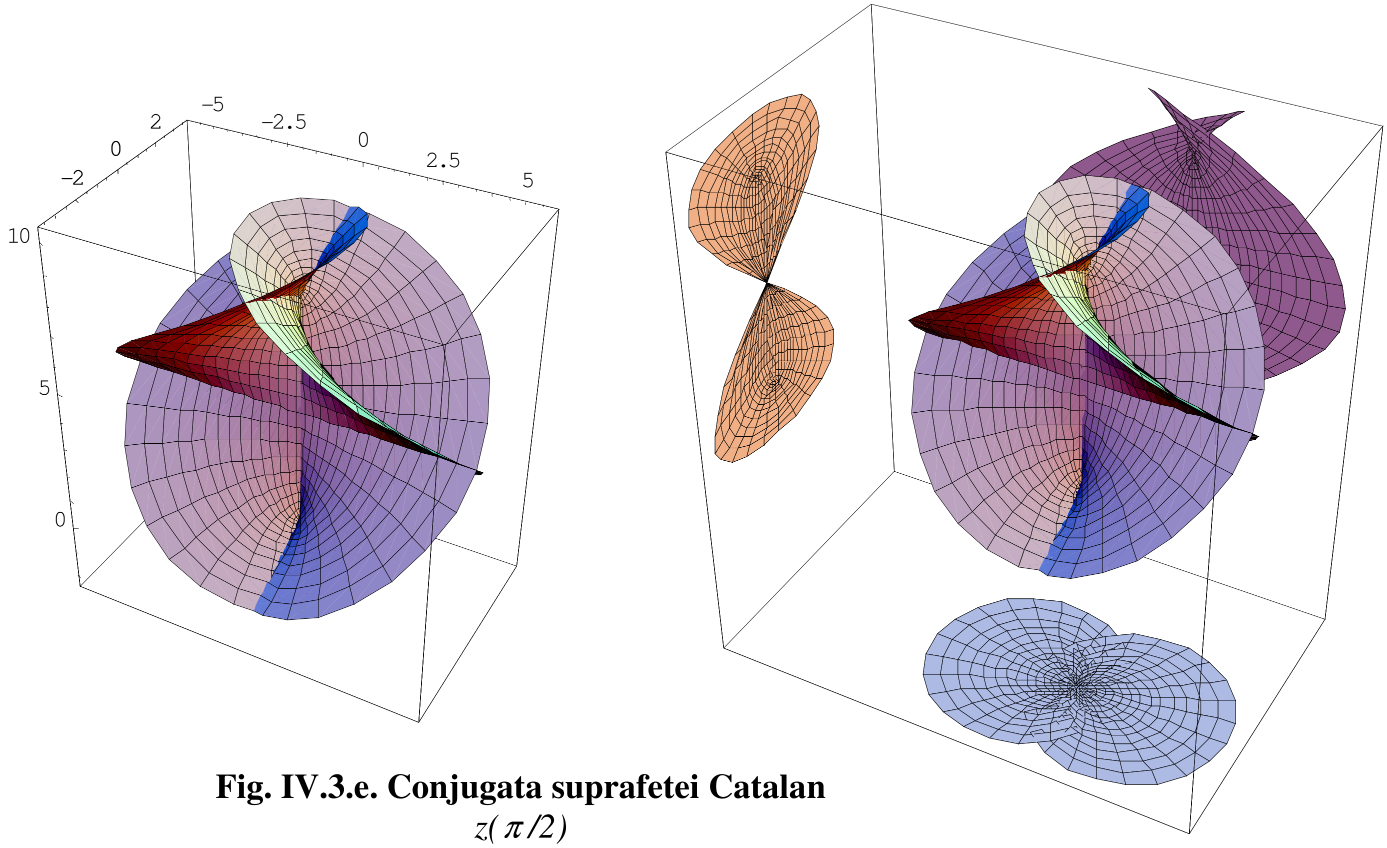


Fig. IV.3.e. Conjugata suprafetei Catalan
$z(\pi/2)$

IV.5.3. Familia asociata suprafetei Enneper

Pentru a gasi familia asociata unei suprafetei Enneper $x(u,v)=\left(u-\frac{u^3}{3}+uv^2,-v+\frac{v^3}{3}-vu^2,u^2-v^2\right)$ aflam mai intai suprafata conjugata $y(u,v)$:

$$y(u,v)=\mathcal{Im}\left(2x\left(\frac{z}{2},\frac{z}{2i}\right)-x(0,0)\right)=\mathcal{Im}\left(z-\frac{z^3}{3},iz+i\frac{z^3}{3},\frac{z^2}{2}\right)=$$

$$=\mathcal{Im}\left(u+iv-\frac{u^3+3iu^2v-3uv^2-iv^3}{3},iu-v+\frac{i(u^3+3iu^2v-3uv^2-iv^3)}{3},u^2+2iuv-v^2\right)=$$

$$=\left(v-u^2v+\frac{v^3}{3},u+\frac{u^3}{3}-uv^2,2uv\right).$$

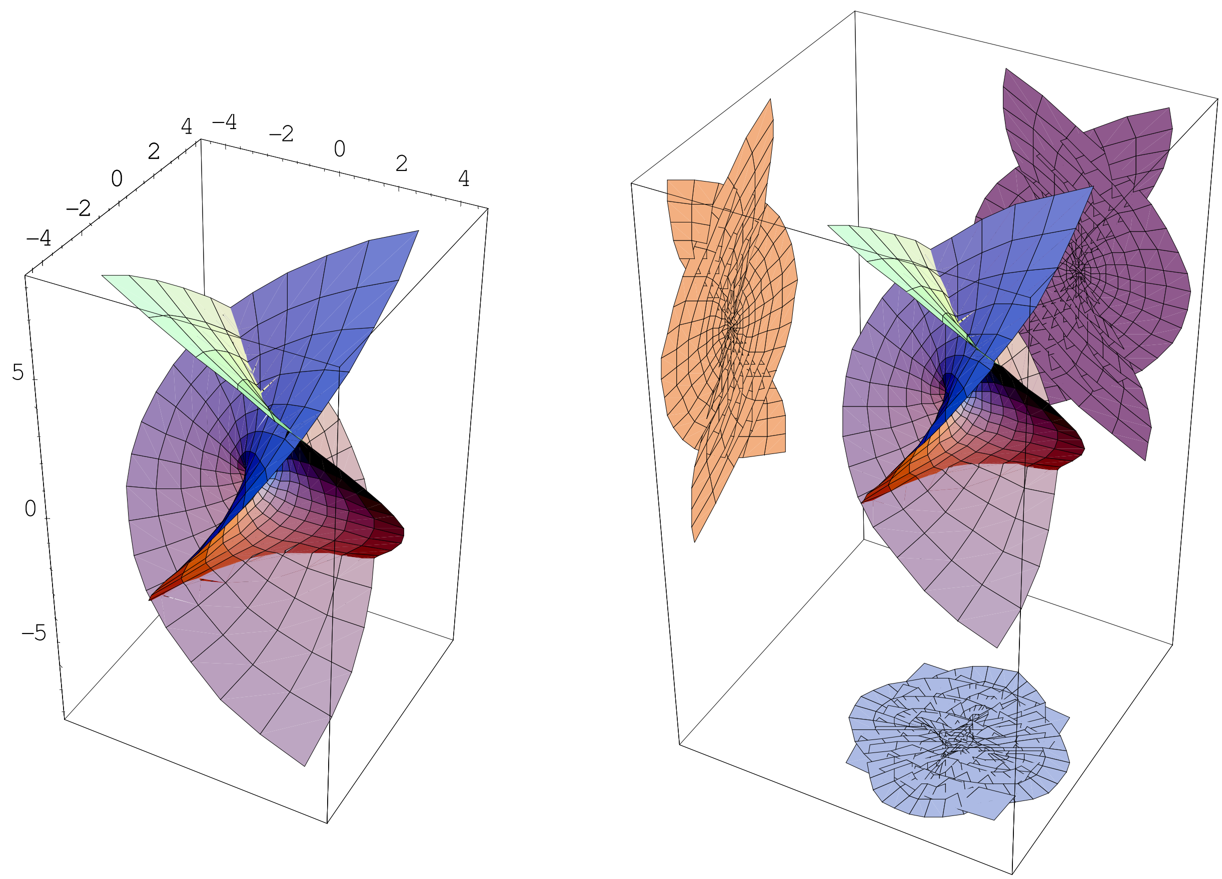


Fig. IV.4. Conjugata suprafetei Enneper

Familia asociata celor doua suprafete conjugate este data de

$$z(t) = cos\ t\left(u-\frac{u^3}{3}+uv^2,-v+\frac{v^3}{3}-vu^2,u^2-v^2\right)+sin\ t\left(v-u^2v+\frac{v^3}{3},u+\frac{u^3}{3}-uv^2,2uv\right)$$

Propozitia IV.22.

Fie $t \mapsto z(t)$ familia asociata a suprafetei Enneper. Atunci *z(t)* este o reparametrizare a suprafetei Enneper rotita cu unghiul $-\frac{t}{2}$ in jurul axei *Oz*.

Demonstratie. Pentru a demonstra aceasta afirmatie folosim coordonatele polare. Vom scrie deci $u=\rho\cos\theta$, $v=\rho\sin\theta$.

Din demonstratia *Propozitiei II.2.* am obtinut scrierea parametrizarii suprafatei Enneper in coordonate polare: $(\rho,\theta) = \left(\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho^2\cos(2\theta)\right)$.

Printr-un calcul asemanator obtinem scrierea suprafetei conjugate:

$$y(\rho,\theta)=\left(\rho\sin\theta-\frac{\rho^3}{3}\cos^2\theta\sin\theta+\frac{\rho^3}{3}\sin^3\theta,\rho\cos\theta+\frac{\rho^3}{3}\cos^2\theta-\rho^3\cos\theta\sin^2\theta,2\rho^2\cos\theta\sin\theta\right)=$$

$$=\left(\rho\sin\theta-\frac{\rho^3}{3}\sin\theta(\cos^2\theta-3\sin^2\theta),\rho\sin\theta+\frac{\rho^3}{3}\cos\theta(\cos^2\theta-3\sin^2\theta),\rho^2\sin(2\theta)\right).$$

Cum $\sin\theta(\cos^2\theta-3\sin^2\theta)=\sin\theta[(\cos^2\theta-\sin^2\theta)-2\sin^2\theta]=\sin\theta\cos(2\theta)-2(\sin\theta\sin\theta)\sin\theta=$

$$=\frac{1}{2}(\sin(3\theta)-\sin(-\theta))+(\cos(2\theta)-1)\sin\theta=\frac{1}{2}(\sin(3\theta)+\sin\theta)+\frac{1}{2}(\sin(3\theta)+\sin\theta)-\sin\theta=$$

$$=\sin(3\theta)\ ;$$

$$\cos\theta(\cos^2\theta-3\sin^2\theta)=\cos\theta[(\cos^2\theta-\sin^2\theta)-2\sin^2\theta]=\cos\theta\cos(2\theta)-2(\cos\theta\sin\theta)\sin\theta=$$

$$=\frac{1}{2}(\cos(3\theta)+\cos(-\theta))+\sin(2\theta)\sin\theta=\frac{1}{2}(\cos(3\theta)+\cos\theta)+\frac{1}{2}(\cos(3\theta)-\cos\theta)=\cos(3\theta),$$

$$y(\rho,\theta)=\left(\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho^2\sin(2\theta)\right).$$

Familia asociata se va scrie

$$z(t) = cos\ t\left(\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho^2\cos(2\theta)\right)+$$

$$sin\ t\left(\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho^2\sin(2\theta)\right),$$

unde $cos\ t\left(\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho^2\cos(2\theta)\right)=$

$$=\left(\frac{1}{2}\rho(\cos(t+\theta)+\cos(t-\theta))-\frac{\rho^3}{3}\frac{1}{2}(\cos(t+3\theta)-\cos(t-3\theta)),\right.$$

$$\frac{1}{2}\rho(\sin(t+\theta)-\sin(t-\theta))-\frac{\rho^3}{3}\frac{1}{2}(\sin(t+3\theta)-\sin(t-3\theta),$$

$$\left.\frac{1}{2}\rho^2(\cos(t+2\theta)+\cos(t-2\theta))\right)$$

$$\mathit{sin\ t}\left(\rho\sin\theta-\frac{\rho^3}{3}\sin(3\theta),\rho\cos\theta-\frac{\rho^3}{3}\cos(3\theta),\rho^2\sin(2\theta)\right)=$$

$$=\left(-\frac{1}{2}\rho(\cos(t+\theta)-\cos(t-\theta))+\frac{\rho^3}{3}\frac{1}{2}(\cos(t+3\theta)-\cos(t-3\theta))-\frac{\rho^3}{3}(\cos(t+\theta)-\cos(t-\theta)),\right.$$

$$\frac{1}{2}\rho(\sin(t+\theta)+\sin(t-\theta))+\frac{\rho^3}{3}\frac{1}{2}(\sin(t+3\theta)-\sin(t-3\theta),$$

$$\left.-\frac{1}{2}\rho^2(\cos(t+2\theta)-\cos(t-2\theta))\right)$$

Deci, din calculele anterioare obtinem

$$z(t)=\left(\rho\cos(t-\theta)-\frac{\rho^3}{3}\cos(t-3\theta),\rho\sin(t+\theta)+\frac{\rho^3}{3}\sin(t-3\theta),\rho^2\cos(t-2\theta)\right).$$

Consideram o reparametrizare a suprafetei Enneper cu $u=\rho\cos\left(\theta-\frac{t}{2}\right)$, $v=\rho\sin\left(\theta-\frac{t}{2}\right)$

$$x'(\rho,\theta)=\left(\rho\cos\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\cos\left(3\left(\theta-\frac{t}{2}\right)\right),\rho\sin\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\sin\left(3\left(\theta-\frac{t}{2}\right)\right),\rho^2\cos\left(2\left(\theta-\frac{t}{2}\right)\right)\right).$$

Rotind suprafata data de parametrizarea *x'* in jurul axei *Oz* cu unghiul $-\frac{t}{2}$ obtinem

$$x''(\rho,\theta)=\left(\left[\rho\cos\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\cos\left(3\left(\theta-\frac{t}{2}\right)\right)\right]\cos\left(-\frac{t}{2}\right)+\left[\rho\sin\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\sin\left(3\left(\theta-\frac{t}{2}\right)\right)\right]\sin\left(-\frac{t}{2}\right),\right.$$

$$\left[\rho\sin\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\sin\left(3\left(\theta-\frac{t}{2}\right)\right)\right]\cos\left(-\frac{t}{2}\right)-\left[\rho\cos\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\cos\left(3\left(\theta-\frac{t}{2}\right)\right)\right]\sin\left(-\frac{t}{2}\right),$$

$$\left.\rho^2\cos\left(2\left(\theta-\frac{t}{2}\right)\right)\right)=$$

$$=\left(\rho\cos\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\cos\left(3\left(\theta-\frac{t}{2}\right)\right),\rho\sin\left(\theta-\frac{t}{2}\right)-\frac{\rho^3}{3}\sin\left(3\left(\theta-\frac{t}{2}\right)\right),\rho^2\cos\left(2\left(\theta-\frac{t}{2}\right)\right)\right)$$

Observam ca expresia suprafata *x''* , obtinuta prin rotirea in jurul axei *Oz* cu unghiul $-\frac{t}{2}$ a suprafetei *x'* , este chiar *z(t)*, de unde concluzia propozitiei.

IV.5.4. Familia asociata suprafetei Henneberg

Pentru a gasi familia asociata unei suprafetei Henneberg $x(u,v) = \left(2shu\cos v - \frac{2}{3}sh(3u)\cos(3v), 2shu\sin v + \frac{2}{3}sh(3u)\sin(3v), 2ch(2u)\cos(2v)\right)$ aflam suprafata conjugata $y(u,v)$:

$$y(u,v) = \mathcal{Im}\left(2x\left(\frac{z}{2},\frac{z}{2i}\right) - x(0,0)\right) =$$

$$= 2\mathcal{Im}\left(2sh\frac{z}{2}\cos\frac{z}{2i} - \frac{2}{3}sh\frac{3z}{2}\cos\frac{3z}{2i}, 2sh\frac{z}{2}\sin\frac{z}{2i} + \frac{2}{3}sh\frac{3z}{2}\sin\frac{3z}{2i}, 2chz\cos\frac{z}{i} - 2\right).$$

Cum

$$sh\frac{z}{2}\cos\frac{z}{2i} = sh\frac{z}{2}\cos\left(-\frac{iz}{2}\right) = sh\frac{z}{2}\cos\frac{iz}{2} = sh\frac{z}{2}ch\frac{z}{2} = \frac{\cos vshu}{2} + i\frac{\sin vchu}{2}$$ [10],

$$sh\frac{z}{2}\sin\frac{z}{2i} = sh\frac{z}{2}\sin\left(-\frac{iz}{2}\right) = -sh\frac{z}{2}\sin\frac{iz}{2} = -ish\frac{z}{2}sh\frac{z}{2} = i\frac{1}{2} - i\frac{\cos vchu}{2} + \frac{\sin vshu}{2},$$

$$chz\cos\frac{z}{i} = chz\cos(-iz) = chz\cos(iz) = chzchz = \frac{1}{2} + \frac{\cos vchu}{2} + i\frac{\sin vshu}{2},$$ atunci

$$y(u,v) = \left(2\sin vchu - \frac{2}{3}\sin(3v)ch(3u), 2 - 2\cos vchu - \frac{2}{3} - \frac{2}{3}\cos(3v)ch(3u), 2\sin(2v)sh(2u)\right).$$

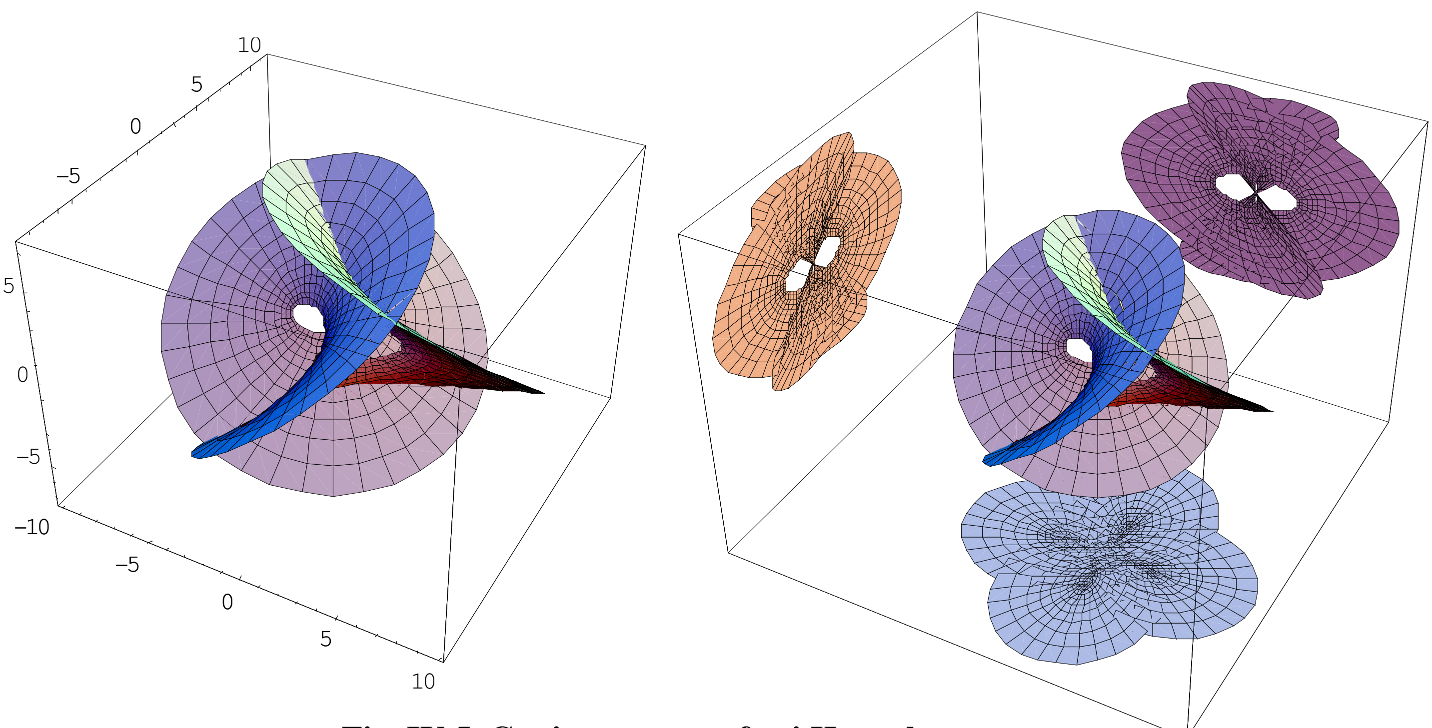


Fig. IV.5. Conjugata suprafetei Henneberg

[10] Calculele complete sunt prezentate in *Anexa 3.2.*

Familia asociata celor doua suprafete conjugate este data de

$$z(t) = \cos t\left(2\,sh u\cos v - \frac{2}{3}sh(3u)\cos(3v), 2\,sh u\sin v + \frac{2}{3}sh(3u)\sin(3v), 2ch(2u)\cos(2v)\right) +$$
$$+\sin t\left(2\sin v\,ch u - \frac{2}{3}\sin(3v)ch(3u), 2 - 2\cos v\,ch u - \frac{2}{3} - \frac{2}{3}\cos(3v)ch(3u), 2\sin(2v)sh(2u)\right)$$

IV.5.5. Familia asociata suprafetei Scherk

Pentru a gasi familia asociata unei suprafetei Scherk $x(u,v) = \left(u, v, \frac{1}{a}\log\left(\frac{\cos(av)}{\cos(au)}\right)\right)$ aflam suprafata sa conjugata:

$$y(u,v) = \mathcal{Im}\left(2x\left(\frac{z}{2}, \frac{z}{2i}\right) - x(0,0)\right) = 2\mathcal{Im}\left(\frac{z}{2}, \frac{z}{2i}, \log\left(\frac{\cos\left(a\frac{z}{2i}\right)}{\cos\left(a\frac{z}{2}\right)}\right)\right).$$

Cum

$\cos\frac{z}{i} = \cos(-iz) = \cos(iz) = chz = \frac{\cos v\,ch u}{2} + i\frac{\sin v\,sh u}{2}$ si $\cos z = ch\,v\cos u + i\,sh\,v\sin u$, obtinem suprafata conjugata:

$$y(u,v) = \left(v, -u, \frac{2}{a}\log\left(\frac{\sin\left(a\frac{v}{2}\right)sh\left(a\frac{u}{2}\right)}{sh\left(a\frac{v}{2}\right)\sin\left(a\frac{u}{2}\right)}\right)\right)$$

Familia asociata celor doua suprafete conjugate este

$$z(t) = \cos t\left(u, v, \frac{1}{a}\log\left(\frac{\cos(av)}{\cos(au)}\right)\right) +$$
$$+\sin t\left(v, -u, \frac{2}{a}\log\left(\frac{\sin\left(a\frac{v}{2}\right)sh\left(a\frac{u}{2}\right)}{sh\left(a\frac{v}{2}\right)\sin\left(a\frac{u}{2}\right)}\right)\right).$$

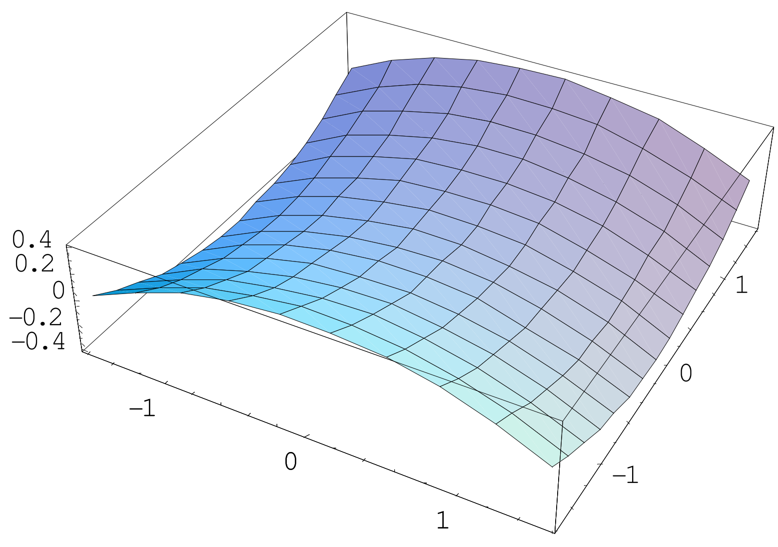


Fig. IV.6. Conjugata suprafetei Scherk

CAPITOLUL V. Reprezentari Weierstrass

In aceasta sectiune vom continua abordarea temei suprafetelor minimale prin intermediul reprezentarii Weierstrass, aceasta permitand investigarea lor intr-o maniera generala si generarea usoara de noi suprafete minimale folosind formula Weierstrass.

V.1. Suprafete minimale prin reprezentarea Weierstrass

Definitie: O functie $f: \mathbb{C} \to \mathbb{C}$ se numeste ***meromorfa*** daca toate punctele sale singulare sunt poli.

Definitie: Fie f si g functii meromorfe definite pe un domeniu $U \subset \mathbb{C}$ si $z_0 \in U$ fixat. Fie

$$\begin{cases} x_1(z) = \mathcal{R}e\left(\int_{z_0}^{z} \frac{f(w)}{2}(1-g(w)^2)dw\right) \\ x_2(z) = \mathcal{R}e\left(\int_{z_0}^{z} \frac{if(w)}{2}(1+g(w)^2)dw\right) \\ x_3(z) = \mathcal{R}e\left(\int_{z_0}^{z} f(w)g(w)dw\right) \end{cases} \quad \text{si} \quad \begin{cases} y_1(z) = \mathcal{I}m\left(\int_{z_0}^{z} \frac{f(w)}{2}(1-g(w)^2)dw\right) \\ y_2(z) = \mathcal{I}m\left(\int_{z_0}^{z} \frac{if(w)}{2}(1+g(w)^2)dw\right) \\ y_3(z) = \mathcal{I}m\left(\int_{z_0}^{z} f(w)g(w)dw\right) \end{cases},$$

unde $z = u+iv$. Cu aceste notatii, $x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ se numeste ***parametrizarea Weierstrass*** determinata de f si g, iar $y(u,v) = (y_1(u,v), y_2(u,v), y_3(u,v))$ ***conjugata parametrizarii Weierstrass***.

Teorema V.1.
Oricare ar fi f functie analitica si g functie meromorfa, parametrizarea Weierstrass si conjugata parametrizarii Weierstrass generate de f si g sunt suprafete minimale izoterme avand metrica data de $ds^2 = \frac{1}{4}|f(z)|^2\left(1+|g(z)|^2\right)^2|dz|^2$.
In particular, x si y sunt izometrice si sunt suprafete regulate, mai putin unde f se anuleaza sau are singularitati.
Demonstratie. Consideram complexificarea parametrizarilor conjugate armonic x si y

$(x+iy)(z) = \int_{z_0}^{z}\left(\frac{f(w)}{2}\left(1-g(w)^2\right), \frac{if(w)}{2}\left(1+g(w)^2\right), f(w)g(w)\right)dw$, care arata ca $(x+iy)(z)$ este un triplet de functii analitice. Cunoastem ca partea reala si partea imaginara a unei functii analitice indeplinesc relatiile Cauchy-Riemann, deci $x_{uu}+ x_{vv} = y_{uu}+ y_{vv} = 0$, adica x si y armonice. Cum x armonic, din *Teorema IV.11 (i),* $\frac{\partial x}{\partial z}$ analitica.
Din *Lema IV.12.* avem ca

$$\frac{\partial x}{\partial z} = \frac{1}{2}\frac{d}{dz}(x+iy) = \frac{1}{2}\left(\frac{f(z)}{2}\left(1-g(z)^2\right), i\frac{f(z)}{2}\left(1+g(z)^2\right), f(z)g(z)\right) = \frac{1}{2}\left(\phi_1(x), \phi_2(x), \phi_3(x)\right)$$

Cum

$$\phi_1(x)^2+\phi_2(x)^2+\phi_3(x)^2 = \left(\frac{f(z)}{2}\left(1-g(z)^2\right)\right)^2 + \left(i\frac{f(z)}{2}\left(1+g(z)^2\right)\right)^2 + f(z)^2g(z)^2 =$$
$$= \frac{1}{4}\left(f(z)^2\left(1-g(z)^2\right)^2 - f(z)^2\left(1+g(z)^2\right)^2 + 4f(z)^2g(z)^2\right) = 0,$$

aplicand *Teorema IV.11.*si *Lema IV.13.* rezulta $(x+iy)(z)$ curba minimala si x,y suprafete minimale izoterme (armonice si izoterme).
Cum x este izoterma, avem $\langle x_u, x_u\rangle = \langle x_v, x_v\rangle$ si, conform *Lemei IV.10.* deducem

$$E = G = 2\sum_{k=1}^{3}|\phi_k(x)|^2 = \frac{1}{8}\left(|f|^2\left|1-g^2\right|^2 + |f|^2\left|1+g^2\right|^2 + 4|fg|^2\right) = \frac{1}{4}|f|^2\left(1+|g|^2\right)^2,$$

de unde $ds^2 = \frac{1}{4}|f|^2\left(1+|g|^2\right)^2(du^2+dv^2) = \frac{1}{4}|f|^2\left(1+|g|^2\right)^2|dz|^2$.

Din rezultatul anterior si *Lema IV.13.* putem deduce urmatorul corolar:

Corolar V.2.
Fie f,g functii meromorfe definite pe domeniul $U\subset\mathbb{C}$, fie x parametrizarea Weierstrass si y conjugata sa determinate de f si g. Atunci $z\mapsto(x+iy)(z)$ este curba minimala.
Reciproc, fie $\psi : U \to \mathbb{C}^3$ o curba minimala, $\psi' = (\varphi_1, \varphi_2, \varphi_3)$. Presupunand ca $\varphi_1 - i\varphi_2$ nu este identic nula, definim $f = \varphi_1 - i\varphi_2$ si $g = \frac{\varphi_3}{\varphi_1 - i\varphi_2}$. Atunci f si g ne dau reprezentarea Weierstrass a curbei minimale ψ: $\psi' = \left(\frac{f}{2}\left(1-g^2\right), \frac{if}{2}\left(1+g^2\right), fg\right)$.

Cunoscand definitia familiei asociate unei parametrizari, din sectiunea IV.2 putem rescrie *Teorema IV.7.* pentru parametrizari Weierstrass astfel:

Teorema V.3.
Fie f,g functii meromorfe. Familia asociata parametrizarii Weierstrass determinata de f si g este data de $t \mapsto z(t) = (z_1(t), z_2(t), z_3(t))$, unde

$$\begin{cases} z_1(t)(z) = \mathcal{R}e\left(e^{-it}\int_{z_0}^{z}\frac{f(w)}{2}(1-g(w)^2)dw\right) \\ z_2(t)(z) = \mathcal{R}e\left(e^{-it}\int_{z_0}^{z}\frac{f(w)}{2}(1+g(w)^2)dw\right). \\ z_3(t)(z) = \mathcal{R}e\left(e^{-it}\int_{z_0}^{z}f(w)g(w)dw\right) \end{cases}$$

Pentru fiecare t, $z(t)$ este parametrizare Weierstrass determinata de functiile meromorfe $e^{-it}f$ si g.

Parametrizarea Weierstrass si conjugata sa fiind izometrice, au aceeasi curbura Gauss, a carei formula este data in urmatorul reultat:

Teorema V.4.
Parametrizarea Weierstrass determinata de functiile meromorfe f,g are curbura Gauss

$$K = \frac{-16|g'|^2}{|f|^2\left(1+|g|^2\right)^4}.$$

Mai mult, formula este aceeasi pentru fiecare membru al familiei asociate de parametrizari Weierstrass determininate de f si g.

Demonstratie. Din *Propozitia IV.2* cunoastem ca $K = \dfrac{-\Delta \ln \lambda}{\lambda^2}$, unde factorul de scalare

$\lambda = \sqrt{< x_u, x_u >} = \sqrt{E} = \frac{1}{2}|f|\left(1+|g|^2\right)$ (din demonstratia *Teoremei V.1*).

$$\Delta \ln|f(z)| = 2\frac{\partial^2}{\partial \bar{z}\partial z}\ln|f(z)|^2 = 2\frac{\partial^2}{\partial \bar{z}\partial z}\left(\ln f(z) + \ln \overline{f(z)}\right) = 0$$

$$\Delta \ln\left(1+|g(z)|^2\right) = 4\frac{\partial^2}{\partial \bar{z}\partial z}\ln\left(1+g(z)\overline{g(z)}\right) = 4\frac{\partial}{\partial \bar{z}}\left(\frac{g'(z)\overline{g(z)}}{1+|g(z)|^2}\right) = \frac{4|g'(z)|^2}{\left(1+|g(z)|^2\right)^2}.$$

Inlocuind in formula curburii Gauss, obtinem

$$K=\frac{-\Delta\ln\left(\frac{1}{2}|f|\left(1+|g|^2\right)\right)}{\frac{1}{4}|f|^2\left(1+|g|^2\right)^2}=\frac{-\Delta\ln\left(\left(1+|g|^2\right)\right)}{\frac{1}{4}|f|^2\left(1+|g|^2\right)^2}=\frac{-\frac{4|g'(z)|^2}{\left(1+|g(z)|^2\right)^2}}{\frac{1}{4}|f|^2\left(1+|g|^2\right)^2}=\frac{-16|g'(z)|^2}{|f|^2\left(1+|g|^2\right)^4}.$$

Ca o consecinta imediata a teoremei anterioare avem urmatorul rezultat:

Corolar V.5.
Curbura Gauss a unei parametrizari Weierstrass determinata de functiile meromorfe *f,g* se anuleaza in zerourile functiei *g'*. In plus, daca *g'* nu este functie identic nula, zerourile curburii Gauss sunt puncte izolate.

V.2. Exemple de parametrizari Weierstrass

V.2.1. Suprafata Enneper

Parametrizarea Weierstass a suprafatei Enneper se obtine prin alegerea $f\equiv 1$, $g(z)=z$. Pentru simplitate, putem alege $z_0=0$. Atunci:

$$x_1(z)=\mathcal{Re}\left(\int_{z_0}^{z}\frac{f(w)}{2}(1-g(w)^2)dw\right)=\mathcal{Re}\left(\int_{z_0}^{z}\frac{1}{2}(1-w^2)dw\right)=\mathcal{Re}\left(\frac{1}{2}\left(z-\frac{z^3}{3}\right)\right)=$$

$$=\mathcal{Re}\left(\frac{1}{2}\left(u+iv-\frac{(u+iv)^3}{3}\right)\right)=\mathcal{Re}\left(\frac{1}{2}\left((u+iv)-\frac{1}{3}\left(u^3+3iu^2v-3uv^2-iv^3\right)\right)\right)=\frac{1}{2}\left(u-\frac{u^3}{3}+uv^2\right)$$

$$x_2(z)=\mathcal{Re}\left(\int_{z_0}^{z}\frac{if(w)}{2}(1+g(w)^2)dw\right)=\mathcal{Re}\left(\int_{z_0}^{z}\frac{i}{2}(1+w^2)dw\right)=\mathcal{Re}\left(\frac{i}{2}\left(z+\frac{z^3}{3}\right)\right)=$$

$$=\mathcal{Re}\left(\frac{i}{2}\left(u+iv+\frac{(u+iv)^3}{3}\right)\right)=\mathcal{Re}\left(\frac{i}{2}\left((u+iv)+\frac{1}{3}\left(u^3+3iu^2v-3uv^2-iv^3\right)\right)\right)=\frac{1}{2}\left(-v-u^2v+\frac{v^3}{3}\right)$$

$$x_3(z) = \mathcal{R}e\left(\int_{z_0}^{z} f(w)g(w)dw\right) = \mathcal{R}e\left(\int_{z_0}^{z} wdw\right) = \mathcal{R}e\left(\frac{z^2}{2}\right) = \mathcal{R}e\left(\frac{1}{2}\left(u^2+2iuv-v^2\right)\right) = \frac{1}{2}\left(u^2-v^2\right)$$

Deci, parametrizarea Enneper este $x(u,v) = \left(\frac{1}{2}\left(u-\frac{u^3}{3}+uv^2\right), \frac{1}{2}\left(-v-u^2v+\frac{v^3}{3}\right), \frac{1}{2}\left(u^2-v^2\right)\right)$ si,

eliminind constanta multiplicativa $\frac{1}{2}$, obtinem exact definitia suprafetei Enneper din sectiunea II.1.

Vom calcula si conjugata Weierstrass a parametrizarii Enneper:

$$y_1(z) = \mathcal{I}m\left(\int_{z_0}^{z} \frac{f(w)}{2}(1-g(w)^2)dw\right) = \mathcal{I}m\left(\frac{1}{2}\left(u+iv-\frac{1}{3}\left(u^3+3iu^2v-3uv^2-iv^3\right)\right)\right) = \frac{1}{2}\left(v-u^2v+\frac{v^3}{3}\right)$$

$$y_2(z) = \mathcal{I}m\left(\int_{z_0}^{z} \frac{if(w)}{2}(1+g(w)^2)dw\right) = \mathcal{I}m\left(\frac{i}{2}\left((u+iv)+\frac{1}{3}\left(u^3+3iu^2v-3uv^2-iv^3\right)\right)\right) = \frac{1}{2}\left(u+\frac{u^3}{3}-uv^2\right)$$

$$y_3(z) = \mathcal{I}m\left(\int_{z_0}^{z} f(w)g(w)dw\right) = \mathcal{I}m\left(\frac{1}{2}\left(u^2+2iuv-v^2\right)\right) = uv\ .$$

Deci, conjugata parametrizarii Enneper este

$y(u,v) = \left(\frac{1}{2}\left(v-u^2v+\frac{v^3}{3}\right), \frac{1}{2}\left(u+\frac{u^3}{3}-uv^2\right), uv\right)$, expresie obtinuta si in sectiunea IV.5.3.

V.2.2. Elicoidul si catenoidul

Catenoidul poate fi obtinut alegand $f(z) = -e^{-z}$ si $g(z) = -e^{z}$. Efectuand calculele, obtinem:

$$x_1(z) = \mathcal{R}e\left(\int_{z_0}^{z} \frac{f(w)}{2}(1-g(w)^2)dw\right) = \mathcal{R}e\left(\int_{z_0}^{z} -\frac{e^{-w}}{2}(1-e^{2w})dw\right) = \mathcal{R}e\left(\frac{1}{2}\left(e^{-z}+e^{z}\right)-1\right) =$$

$$= \mathcal{R}e\left(\frac{1}{2}\left(e^{-u}(\cos v - i\sin v)+e^{u}(\cos v + i\sin v)\right)-1\right) = \cos v\frac{e^u+e^{-u}}{2}-1 = \cos v\, chu - 1\ ,$$

$$x_2(z) = \mathcal{R}e\left(\int_{z_0}^{z} \frac{if(w)}{2}(1+g(w)^2)dw\right) = \mathcal{R}e\left(\int_{z_0}^{z} -\frac{ie^{-w}}{2}(1+e^{2w})dw\right) = \mathcal{R}e\left(\frac{i}{2}\left(e^{-z}-e^{z}\right)+i\right) =$$

$$= \mathcal{R}e\left(\frac{i}{2}\left(e^{-u}(\cos v - i\sin v)-e^{u}(\cos v + i\sin v)\right)+i\right) = \sin v\frac{e^u+e^{-u}}{2} = \sin v\, chu\ ,$$

$$x_3(z) = \mathcal{Re}\left(\int_{z_0}^{z} f(w)g(w)dw\right) = \mathcal{Re}\left(\int_{z_0}^{z} (-e^{-z})(-e^{z})dw\right) = \mathcal{Re}\ z = u\ .$$

Deci, catenoidul este dat de parametrizarea $x(u,v) = (\cos v ch u - 1, \sin v ch u, u)$, care, prin translatia cu o unitate pe axa Ox si interschimbarea variabilelor u,v conduce la definitia din sectiunea II.3.
Cu calcule similare, obtinem conjugata Weierstrass:

$$y_1(z) = \mathcal{Im}\left(\int_{z_0}^{z} \frac{f(w)}{2}(1-g(w)^2)dw\right) =$$

$$= \mathcal{Im}\left(\frac{1}{2}\left(e^{-u}(\cos v - i\sin v) + e^{u}(\cos v + i\sin v)\right) - 1\right) = \sin v\frac{e^{u}-e^{-u}}{2} = \sin v sh u\ ,$$

$$y_2(z) = \mathcal{Im}\left(\int_{z_0}^{z} \frac{if(w)}{2}(1+g(w)^2)dw\right) =$$

$$= \mathcal{Im}\left(\frac{i}{2}\left(e^{-u}(\cos v - i\sin v) - e^{u}(\cos v + i\sin v)\right) + i\right) = -\cos v\frac{e^{u}-e^{-u}}{2} = -\cos v sh u\ ,$$

$$y_3(z) = \mathcal{Im}\left(\int_{z_0}^{z} f(w)g(w)dw\right) = \mathcal{Im}\left(\int_{z_0}^{z} (-e^{-z})(-e^{z})dw\right) = \mathcal{Im}\ z = v\ ,$$

deci *y(u,v) = (sinv shu, -cosv shu, v)*, suprafata ce reprezinta o reparametrizare a elicoidului, a carui definitie e data in sectiunea II.2.

V.2.3. Suprafata Henneberg

Calculand parametrizarea Weierestrass pentru alegerea $f(z) = 1 - \frac{1}{z^4}$ si $g(z) = z$ vom obtine o reparametrizare a suprafetei Henneberg.

$$x_1(z) = \mathcal{Re}\left(\int_{z_0}^{z} \frac{f(w)}{2}(1-g(w)^2)dw\right) = \mathcal{Re}\left(\int_{z_0}^{z} \frac{1}{2}\left(1-\frac{1}{w^4}\right)(1-w^2)dw\right) =$$

$$= \mathcal{Re}\left(\frac{1}{2}\left(z - \frac{z^3}{3} + \frac{1}{3z^3} - \frac{1}{z}\right)\right) = \mathcal{Re}\left(\frac{1}{2}\left(z - \frac{z^3}{3} + \frac{\bar{z}^3}{3|z|^6} - \frac{\bar{z}}{|z|^2}\right)\right) =$$

$$= \mathcal{Re}\left(\frac{1}{2}\left(u+iv - \frac{u^3+3u^2iv-3uv^2-iv^3}{3} + \frac{u^3-3u^2iv-3uv^2+iv^3}{3(u^2+v^2)^3} - \frac{u-iv}{u^2+v^2}\right)\right) =$$

$$= \frac{1}{2}\left[u\left(1-\frac{1}{u^2+v^2}\right)-\frac{u^3-3uv^2}{3}\left(1-\frac{1}{\left(u^2+v^2\right)^3}\right)\right] = \frac{1}{2}\left[r\cos t\left(1-\frac{1}{r^2}\right)-\frac{r^3\cos 3t}{3}\left(1-\frac{1}{r^6}\right)\right] =$$

$$= \frac{1}{2}\left(\frac{\cos t(r^2-1)}{r}-\frac{\cos 3t(r^6-1)}{3r^3}\right)$$

unde $z = u+iv = r\cos t + i\, r\sin t$,

$$x_2(z) = \mathcal{R}e\left(\int_{z_0}^{z}\frac{if(w)}{2}(1+g(w)^2)dw\right) = \mathcal{R}e\left(\int_{z_0}^{z}\frac{i}{2}\left(1-\frac{1}{w^4}\right)(1+w^2)dw\right) =$$

$$= \mathcal{R}e\left(\frac{i}{2}\left(z+\frac{z^3}{3}+\frac{1}{3z^3}+\frac{1}{z}\right)\right) = \mathcal{R}e\left(\frac{1}{2}\left(z+\frac{z^3}{3}+\frac{\bar{z}^3}{3|z|^6}+\frac{\bar{z}}{|z|^2}\right)\right) =$$

$$= \mathcal{R}e\left(\frac{i}{2}\left(u+iv+\frac{u^3+3u^2iv-3uv^2-iv^3}{3}+\frac{u^3-3u^2iv-3uv^2+iv^3}{3(u^2+v^2)^3}+\frac{u-iv}{u^2+v^2}\right)\right) =$$

$$= \frac{1}{2}\left[-v\left(1-\frac{1}{u^2+v^2}\right)-\frac{v^3-3u^2v}{3}\left(1-\frac{1}{\left(u^2+v^2\right)^3}\right)\right] = \frac{1}{2}\left[-r\sin t\left(1-\frac{1}{r^2}\right)-\frac{r^3\sin 3t}{3}\left(1-\frac{1}{r^6}\right)\right] =$$

$$= -\frac{1}{2}\left(\frac{\sin t(r^2-1)}{r}+\frac{\sin 3t(r^6-1)}{3r^3}\right),$$

$$x_3(z) = \mathcal{R}e\left(\int_{z_0}^{z}f(w)g(w)dw\right) = \mathcal{R}e\left(\int_{z_0}^{z}\left(1-\frac{1}{w^4}\right)wdw\right) = \mathcal{R}e\left(\frac{z^2}{2}+\frac{1}{2z^2}\right) = \mathcal{R}e\left(\frac{z^2}{2}+\frac{\bar{z}^2}{2|z|^4}\right) =$$

$$= \mathcal{R}e\left(\frac{u^2+2uiv-v^2}{2}+\frac{u^2-2uiv-v^2}{2(u^2+v^2)^2}\right) = \frac{u^2-v^2}{2}\left(1+\frac{1}{\left(u^2+v^2\right)^2}\right) = \frac{\cos(2t)(r^4+1)}{2r^2}.$$

Am obtinut deci, suprafata

$$x(r,t) = \left(\frac{1}{2}\left(\frac{\cos t(r^2-1)}{r}-\frac{\cos 3t(r^6-1)}{3r^3}\right), -\frac{1}{2}\left(\frac{\sin t(r^2-1)}{r}+\frac{\sin 3t(r^6-1)}{3r^3}\right), \frac{\cos(2t)(r^4+1)}{2r^2}\right),$$

ce reprezinta o reparametrizare a suprafetei Henneberg definita in sectiunea II.5.

V.2.4. Suprafata Bour

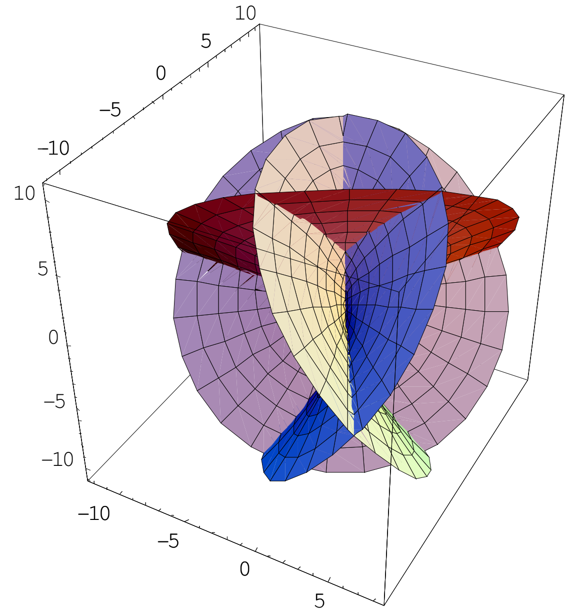


Fig. IV.7. Suprafata Bour

Parametrizarea Weierstrass corespunzatoare alegerii $f\equiv 1$, $g(z)=\sqrt{z}$ se numeste suprafata Bour:

$$x_1(z)=\mathcal{R}e\left(\int_{z_0}^{z}\frac{f(w)}{2}(1-g(w)^2)dw\right)=$$

$$=\mathcal{R}e\left(\int_{z_0}^{z}\frac{1}{2}(1-w)dw\right)=\mathcal{R}e\frac{1}{2}\left(z-\frac{z^2}{2}\right)=$$

$$=\mathcal{R}e\left(\frac{1}{2}(u+iv)-\frac{u^2+2uiv-v^2}{4}\right)=$$

$$=\frac{1}{2}r\cos t-\frac{1}{4}r^2\cos(2t),$$

$$x_2(z)=\mathcal{R}e\left(\int_{z_0}^{z}\frac{if(w)}{2}(1+g(w)^2)dw\right)=\mathcal{R}e\left(\int_{z_0}^{z}\frac{i}{2}(1+w)dw\right)=\mathcal{R}e\frac{i}{2}\left(z+\frac{z^2}{2}\right)=$$

$$=\mathcal{R}e\left(i\left(\frac{1}{2}(u+iv)+\frac{u^2+2uiv-v^2}{4}\right)\right)=-\frac{1}{2}r\cos t-\frac{1}{4}r^2\sin(2t),$$

$$x_3(z)=\mathcal{R}e\left(\int_{z_0}^{z}f(w)g(w)dw\right)=\mathcal{R}e\left(\int_{z_0}^{z}\sqrt{w}dw\right)=\mathcal{R}e\left(\frac{2}{3}z^{3/2}\right)=\frac{2}{3}(u^3-3uv^2)^{3/2}=\frac{2}{3}r^{3/2}\left(4\cos^3 t-3\cos t\right)=$$

$$=\frac{2}{3}r^{3/2}\cos(3t).$$

Am obtinut, deci, suprafata Bour: $x(r,t)=\left(\frac{1}{2}r\cos t-\frac{1}{4}r^2\cos(2t),-\frac{1}{2}r\cos t-\frac{1}{4}r^2\sin(2t),\frac{2}{3}r^{3/2}\cos(3t)\right)$

V.2.4. Suprafete cu terminatie plana. Suprafata Richmond

O familie interesanta de suprafete minimale este generata prin alegerea functiilor meromorfe f,g pentru reprezentarea Weierstrass astfel: $f(z)=\frac{1}{z^2}$ si $g(z)=z^{n+1}$.

$$x_1(z)=\mathcal{R}e\left(\int_{z_0}^{z}\frac{f(w)}{2}(1-g(w)^2)dw\right)=\mathcal{R}e\left(\int_{z_0}^{z}\frac{1}{2w^2}(1-w^{2n+2})dw\right)=\mathcal{R}e\frac{1}{2}\left(-\frac{1}{z}-\frac{z^{2n+1}}{2n+1}\right)=\mathcal{R}e\frac{1}{2}\left(-\frac{\bar{z}}{|z|^2}-\frac{z^{2n+1}}{2n+1}\right)$$

$$= \mathcal{Re}\left(\frac{1}{2}\left(-\frac{r(\cos t - i\sin t)}{r^2} - \frac{r^{2n+1}(\cos((2n+1)t)+i\sin((2n+1)t))}{2n+1}\right)\right) = \frac{1}{2}\left(-\frac{\cos t}{r} - \frac{r^{2n+1}\cos((2n+1)t)}{2n+1}\right)$$

$$x_2(z) = \mathcal{Re}\left(\int_{z_0}^{z}\frac{if(w)}{2}(1+g(w)^2)dw\right) = \mathcal{Re}\left(\int_{z_0}^{z}\frac{i}{2w^2}(1+w^{2n+2})dw\right) = \mathcal{Re}\left(\frac{i}{2}\left(-\frac{1}{z}+\frac{z^{2n+1}}{2n+1}\right)\right) = \mathcal{Re}\left(\frac{i}{2}\left(-\frac{\bar{z}}{|z|^2}+\frac{z^{2n+1}}{2n+1}\right)\right)$$

$$= \mathcal{Re}\left(\frac{i}{2}\left(-\frac{r(\cos t - i\sin t)}{r^2}+\frac{r^{2n+1}(\cos((2n+1)t)+i\sin((2n+1)t))}{2n+1}\right)\right) = \frac{1}{2}\left(-\frac{\sin t}{r}+\frac{r^{2n+1}\sin((2n+1)t)}{2n+1}\right)$$

$$x_3(z) = \mathcal{Re}\left(\int_{z_0}^{z} f(w)g(w)dw\right) = \mathcal{Re}\left(\int_{z_0}^{z}\frac{1}{w^2}w^{n+1}dw\right) = \mathcal{Re}\left(\frac{z^n}{n}\right) = \mathcal{Re}\left(\frac{r^n(\cos(nt)+i\sin(nt))}{n}\right) = \frac{r^n(\cos(nt)}{n}$$

Deci, se obtin suprafetele

$$x(r,t)(n) = \left(\frac{1}{2}\left(-\frac{\cos t}{r} - \frac{r^{2n+1}\cos((2n+1)t)}{2n+1}\right), \frac{1}{2}\left(-\frac{\sin t}{r} + \frac{r^{2n+1}\sin((2n+1)t)}{2n+1}\right), \frac{r^n(\cos(nt)}{n}\right).$$

Pentru $n = 1$ se obtine suprafata minimala Richmond:

$$x(r,t) = \left(\frac{1}{2}\left(-\frac{\cos t}{r} - \frac{r^3\cos(3t)}{3}\right), \frac{1}{2}\left(-\frac{\sin t}{r} + \frac{r^3\sin(3t)}{3}\right), r\cos t\right)$$

$$= \left(\frac{1}{2}\left(-\frac{r\cos t}{r^2} - \frac{r^3\cos(3t)}{3}\right), \frac{1}{2}\left(-\frac{r\sin t}{r^2} + \frac{r^3\sin(3t)}{3}\right), u\right)$$

$$= \left(\frac{1}{2}\left(-\frac{u}{u^2+v^2} - \frac{u^3-3uv^2}{3}\right), \frac{1}{2}\left(-\frac{v}{u^2+v^2} - \frac{v^3-3uv^2}{3}\right), u\right)$$

$$= \left(\frac{-3u-u^5+2u^3v^2+3uv^4}{6(u^2+v^2)}, \frac{-3v+v^5-2u^2v^3+3u^4v}{6(u^2+v^2)}, u\right)$$

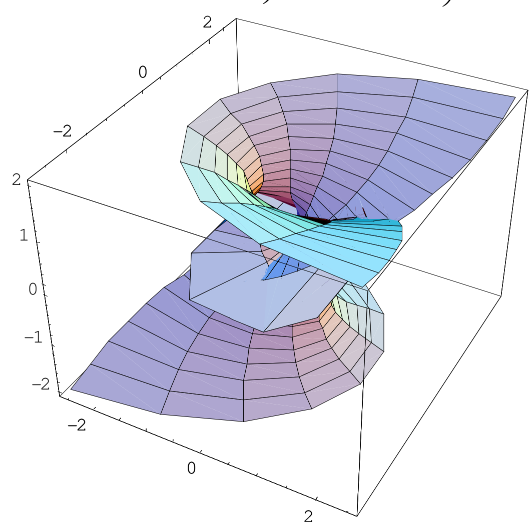
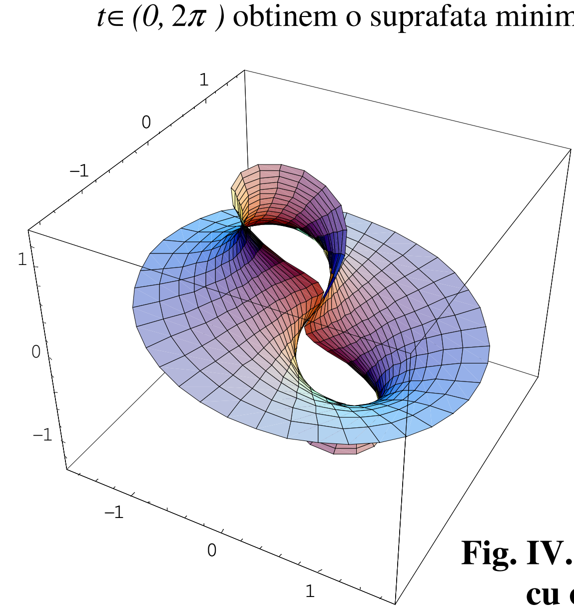


Fig. IV.8. Suprafata Richmond

Iar pentru o alegere a parametrilor: $n = 1$, $r \in (0{,}3; 1{,}3)$, $t \in (0, 2\pi)$ obtinem o suprafata minimala cu terminatie plana.



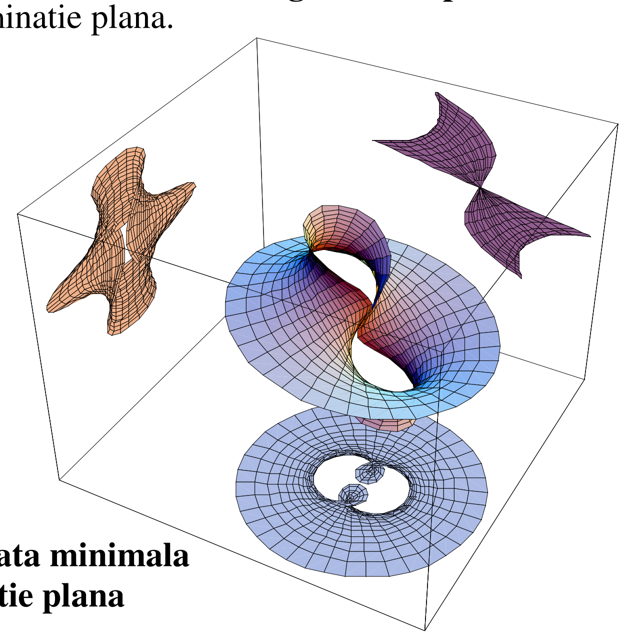


Fig. IV.9. Suprafata minimala cu o terminatie plana

V.2.6. Suprafata Costa

Pentru a descrie aceasta suprafata vom introduce ***functiile Weierstrass*** $\wp$ si ζ .

Fie w_1 , $w_2 \in \mathbb{C}$, $w_1 \neq 0$, $w_2 \neq 0$ astfel incat $\mathfrak{Im}\left(\frac{w2}{w1}\right) \geq 0$ – geometric, acest lucru inseamna ca unghiul format de w_1 si w_2 este ascutit. Definim o latice $L \subset \mathbb{C}$ prin $L = \{ m w_1 + n w_2 \mid m,n \in \mathbb{Z}\}$ si $g_2 = 60 \sum\limits_{\substack{w\in L \\ w\neq 0}} \frac{1}{w^4}$ si $g_3 = 140 \sum\limits_{\substack{w\in L \\ w\neq 0}} \frac{1}{w^6}$.

Definitie: ***Functia Weierstrass*** $\wp$ asociata laticii L este definita prin

$$\wp(z,\{g_2 , g_3 \}) = \frac{1}{z^2} + \sum_{\substack{w\in L \\ w\neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Aplicatia $z \mapsto \wp(z,\{ g_2 , g_3 \})$ are proprietatea de a fi dublu periodica, de perioade w_1 si w_2.

Definitie: Multimea $FPP(\{g_2 , g_3 \}) = \{ z\in \mathbb{C} \mid z = \alpha\ w_1 + \beta\ w_2 ,\ 0\leq \alpha , \beta < 1 \}$ se numeste ***paralelogramul de perioada fundamentala*** al functiei $\wp(z,\{g_2 , g_3 \})$.

Observatii:
1. Este important de observat ca imaginea multimii *FPP* prin functia Weierstrass $\wp$ este o translatie de forma $FPP(\{g_2 , g_3 \}) + m w_1 + n w_2$, $m,n \in \mathbb{Z}$.
2. Functia Weierstrass $\wp$ verifica urmatoarea formula:

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)} \right)^2 - \wp(z_1) - \wp(z_2) .$$

Definitie: ***Functia Weierstrass*** ζ este definita prin $\zeta(z,\{g_2 , g_3 \}) = \frac{1}{z} + \sum\limits_{\substack{w\in L \\ w\neq 0}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right)$.

Atunci cand presupunem cunoscute g_2 si g_3 , putem scrie $\wp(z,\{ g_2 , g_3 \}) = \wp(z)$ si $\zeta(z,\{ g_2 , g_3 \}) = \zeta(z)$.

Functia Weierstrass $\wp$ verifica ecuatia diferentiala $\wp'(z)^2 = 4\wp(z)^4 - g_2\ \wp(z) - g_3$. In plus, legatura intre cele doua functii Weierstrass este data de $\zeta'(z) = -\wp(z)$.

Cum functia $\wp'(z)$ este impara si periodica, de perioada w_1, avem $\wp'\left(\frac{w_1}{2}\right) = -\wp'\left(-\frac{w_1}{2}\right) = -\wp'\left(-\frac{w_1}{2}+w_1\right) = -\wp'\left(\frac{w_1}{2}\right)$, de unde rezulta ca $\wp'\left(\frac{w_1}{2}\right) = 0$. Analog, $\wp'\left(\frac{w_2}{2}\right) = \wp'\left(\frac{w_1+w_2}{2}\right) = 0$. Mai mult, singurele zerouri ale funtiei $\wp'$ in multimea *FPP* sunt $\frac{w_1}{2}, \frac{w_2}{2}, \frac{w_1+w_2}{2}$, radacini si pentru polinomul $4\wp(z)^4 - g_2\,\wp(z) - g_3$.

Daca notam $e_1 = \wp\left(\frac{w_1}{2}\right)$, $e_2 = \wp\left(\frac{w_2}{2}\right)$, $e_3 = \wp\left(\frac{w_1+w_2}{2}\right)$, putem scrie $\wp'(z)^2 = 4\wp(z)^4 - g_2\,\wp(z) - g_3 = 4\,(\wp(z) - e_1)\,(\wp(z) - e_2)\,(\wp(z) - e_3)$, unde e_1, e_2, e_3 distincte doua cate doua si verifica relatiile $\begin{cases} e_1 + e_3 + e_3 = 0 \\ e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4} \\ e_1e_2e_3 = \frac{g_3}{4} \end{cases}$.

Functia ζ nu este periodica, dar se poate deduce ca

$$\zeta\,(z+m\,w_1+n\,w_2) = \zeta\,(z) + 2\,m\,\zeta\left(\frac{w_1}{2}\right) + 2\,n\,\zeta\left(\frac{w_2}{2}\right),\quad m,n \in \mathbb{Z},$$

ceea ce implica $\zeta\left(\frac{w_1+w_2}{2}\right) = \zeta\left(\frac{w_1}{2}\right) + \zeta\left(\frac{w_2}{2}\right)$ (pentru $z = 0$, $m = n = \frac{1}{2}$).

Pentru a putea ajunge la o parametrizare a suprafetei Costa vom folosi aceste doua functii Weierstrass. In plus, vom avea nevoie de ***relatia Legendre***: $\zeta\left(\frac{w_1}{2}\right)w_2 - \zeta\left(\frac{w_1}{2}\right)w_1 = \pi$.

Putem considera cazul in care paralelogramul de perioada fundamentala este un patrat. Sa presupunem ca patratul este $\{\, z \in \mathbb{C} \mid 0 \le \mathfrak{Re}\,(z) < 1$ si $0 \le \mathfrak{Im}(z) < 1 \,\}$. Functia Weierstras $\wp$ corespunzatoare acestui patrat este data de valorile $w_1 = 1$ si $w_1 = i$:

$\wp(z, \{g_2, g_3\}) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in Z^2 \\ (m,n)\neq(0,0)}} \left(\frac{1}{(z-m-ni)^2} - \frac{1}{(m+ni)^2} \right)$. Iar cealalta functie Weierstrass pentru patratul ales este $\zeta\,(z, \{g_2, g_3\}) = \frac{1}{z} + \sum_{\substack{(m,n)\in Z^2 \\ (m,n)\neq(0,0)}} \left(\frac{1}{z-m-ni} + \frac{1}{m+ni} + \frac{z}{(m+ni)^2} \right)$.

Putem determina g_2 si g_3 calculand

$e_1+e_2 = \wp\left(\frac{w_1}{2},\{g_2,g_3\}\right) + \wp\left(\frac{w_2}{2},\{g_2,g_3\}\right) = \wp\left(\frac{1}{2},\{g_2,g_3\}\right) + \wp\left(\frac{i}{2},\{g_2,g_3\}\right) = 0$, de unde rezulta ca cele trei relatii intre e_1, e_2, e_3 devin $\begin{cases} e_3 = 0 \\ -4e_1e_2 = g_2 = 4e_1^2 \\ g_3 = 0 \end{cases}$, iar ecuatia diferentiala verificata de $\wp$ devine $\wp'(z)^2 = 4\wp(z)^4 - g_2\,\wp(z) - g_3 = 4\wp(z)(\wp(z)^2 - e_1^2)$.

Pentru a simplifica, se introduc notatiile pentru functiile Weierstrass $\wp$ si ζ corespunzatoare patratului ales [11]: $\begin{cases} P(z) = \wp(z,\{c,0\}) \\ Z(z) = \zeta(z,\{c,0\}) \end{cases}$. Cu aceste notatii avem urmatorul rezultat:

Lema V.6.
Pentru orice $z \in \mathbb{C}$ avem relatiile

$$P\left(z-\frac{1}{2}\right) - P\left(z-\frac{i}{2}\right) - 2e_1 = \frac{16e_1^3 P(z)}{P'(z)^2},\quad iZ(iz) = Z(z),\quad Z\left(\frac{1}{2}\right) = iZ\left(\frac{i}{2}\right) = \frac{\pi}{2},\quad Z\left(\frac{1+i}{2}\right) = \frac{(1-i)\pi}{2}.$$

Demonstratie. Din formula $\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2)$, prin inlocuire directa obtinem

$$P\left(z-\frac{1}{2}\right) = \frac{P'(z)}{4(P(z)-e_1)} - P(z) - e_1 = \frac{P(z)(P(z)^2-e_1^2)}{(P(z)-e_1)^2} - P(z) - e_1 = \frac{P(z)(P(z)+e_1)}{P(z)-e_1} - P(z) - e_1 =$$

$$= \frac{P(z)(P(z)+e_1) - P(z)^2 + e_1^2}{P(z)-e_1} = \frac{e_1P(z)+e_1^2}{P(z)-e_1} = e_1 + \frac{2e_1^2}{P(z)-e_1}.$$

Analog,

$$P\left(z-\frac{i}{2}\right) = \frac{P'(z)}{4(P(z)-e_2)} - P(z) - e_2 = \frac{P(z)(P(z)^2-e_2^2)}{(P(z)-e_2)^2} - P(z) - e_2 = \frac{P(z)(P(z)+e_2)}{P(z)-e_2} - P(z) - e_2 =$$

$$= \frac{P(z)(P(z)+e_2) - P(z)^2 + e_2^2}{P(z)-e_2} = \frac{e_2P(z)+e_2^2}{P(z)-e_2} = e_2 + \frac{2e_2^2}{P(z)-e_2} = -e_1 + \frac{2e_1^2}{P(z)+e_1}.$$

Obtinem, deci

$$P\left(z-\frac{1}{2}\right) - P\left(z-\frac{i}{2}\right) - 2e_1 = \frac{2e_1^2}{P(z)-e_1} - \frac{2e_1^2}{P(z)+e_1} = \frac{4e_1^3}{P(z)^2+e_1^2}.$$

Si cum, in general, $\wp'(z)^2 = 4\wp(z)(\wp(z)^2 - e_1^2)$, rezulta prima relatie a lemei $P\left(z-\frac{1}{2}\right) - P\left(z-\frac{i}{2}\right) - 2e_1 = \frac{16e_1^3 P(z)}{P'(z)^2}$. Cea de-a doua relatie rezulta direct din definitia functiei ζ

[11] Prin calcul se obtine $e_1 \approx 6.87519$, $c \approx 189.0727201$

pentru patrat. Pentru cea de-a treia luam $z = \dfrac{1}{2}$ si obtinem $Z\left(\dfrac{1}{2}\right) = iZ\left(\dfrac{i}{2}\right)$, iar din relatia Legendre pentru $w_1 = 1$ si $w_2 = i$ deducem exact relatia dorita. Cum $\zeta\left(\dfrac{w_1+w_2}{2}\right) = \zeta\left(\dfrac{w_1}{2}\right) + \zeta\left(\dfrac{w_2}{2}\right)$, efectuand aceeasi alegere, $w_1 = 1$ si $w_2 = i$, obtinem si ultima relatie $Z\left(\dfrac{1+i}{2}\right) = Z\left(\dfrac{1}{2}\right) + Z\left(\dfrac{i}{2}\right) = \dfrac{\pi}{2} + \dfrac{\pi}{2i} = \dfrac{(1-i)\pi}{2}$.

Suprafata minimala Costa poate fi definita ca parametrizarea Weierstrass corespunzatoare alegerii functiilor $f = P(z)$ si $g(z)= \dfrac{A}{P'(z)}$, unde $A=\sqrt{2\pi g_2} = 2e_1\sqrt{2\pi} \approx 34.46707$.

Definitie: Curba minimala $C(z)$ definita ca antiderivata curbei

$$C'(z) = \frac{1}{2}\left(\frac{f(z)}{2}\left(1-g(z)^2\right), i\,\frac{f(z)}{2}\left(1+g(z)^2\right), f(z)g(z)\right),$$

cu normalizarea $C\left(\dfrac{1+i}{2}\right) = (0,0,0)$, se numeste ***curba minimala Costa***.

Vom folosi functia Z pentru a exprima curba minimala Costa $C(z)$ fara a folosi integrale. Avem urmatorul rezultat:

Teorema V.7.
Curba minimala Costa este data de $C(z) = (C_1(z), C_2(z), C_3(z))$, unde

$$\begin{cases} C_1(z) = \dfrac{1}{2}\left(-Z(z) + \pi z - i\pi + \dfrac{\pi^2(1+i)}{4e_1} + \dfrac{\pi}{2e_1}\left(Z\left(z-\dfrac{1}{2}\right) - Z\left(z-\dfrac{i}{2}\right)\right)\right) \\ C_2(z) = \dfrac{i}{2}\left(-Z(z) - \pi z + \pi + \dfrac{\pi^2(1+i)}{4e_1} - \dfrac{\pi}{2e_1}\left(Z\left(z-\dfrac{1}{2}\right) - Z\left(z-\dfrac{i}{2}\right)\right)\right) \\ C_3(z) = \dfrac{\sqrt{2\pi}}{4}\left(\ln\left(\dfrac{P(z)-e_1}{P(z)+e_1}\right) - \pi i\right) \end{cases}.$$

Demonstratie. Efectuam calculele pentru prima componenta a curbei minimale Costa: $\dfrac{1}{2}f(w)\left(1-g(w)^2\right) = \dfrac{1}{2}\left(P(w) - \dfrac{A^2 P(w)}{P'(w)^2}\right)$. Din prima relatie din *Lema V.6.* avem $\dfrac{P(w)}{P'(w)^2} = \dfrac{1}{16e_1^3}\left(P\left(w-\dfrac{1}{2}\right) - P\left(w-\dfrac{i}{2}\right) - 2e_1\right)$ si, folosind definitia lui A, rezulta:

$$\frac{1}{2}f(w)\left(1-g(w)^2\right)=\frac{1}{2}\left(P(w)-\frac{A^2}{16e_1^3}\left(P\left(w-\frac{1}{2}\right)-P\left(w-\frac{i}{2}\right)-2e_1\right)\right)=\frac{1}{2}\left(P(w)-\frac{\pi}{2e_1}\left(P\left(w-\frac{1}{2}\right)-P\left(w-\frac{i}{2}\right)-2e_1\right)\right)$$

$$=\frac{1}{2}\left(P(w)+\pi-\frac{\pi}{2e_1}\left(P\left(w-\frac{1}{2}\right)-P\left(w-\frac{i}{2}\right)\right)\right).$$

Prin integrarea ambelor parti obtinem

$$C_1(z)=\int_{(1+i)/2}^{z}\frac{1}{2}f(w)\left(1-g(w)^2\right)dw=\frac{1}{2}\int_{(1+i)/2}^{z}\left(P(w)+\pi-\frac{\pi}{2e_1}\left(P\left(w-\frac{1}{2}\right)-P\left(w-\frac{i}{2}\right)\right)\right).$$

Folosim relatia $\zeta'(z)=-\wp(z)$ si *Lema V.6.* si rezulta:

$$C_1(z)=\frac{1}{2}\left(-Z(w)+\pi w+\frac{\pi}{2e_1}\left(Z\left(w-\frac{1}{2}\right)-Z\left(w-\frac{i}{2}\right)\right)\right)\Bigg|_{(1+i)/2}^{z}=$$

$$=\frac{1}{2}\left(-Z(z)+\pi z+\frac{\pi}{2e_1}\left(Z\left(z-\frac{1}{2}\right)-Z\left(z-\frac{i}{2}\right)\right)+Z\left(\frac{1+i}{2}\right)-\frac{\pi(1+i)}{2}-\frac{\pi}{2e_1}\left(Z\left(\frac{i}{2}\right)-Z\left(\frac{1}{2}\right)\right)\right)=$$

$$=\frac{1}{2}\left(-Z(z)+\pi z+\frac{\pi}{2e_1}\left(Z\left(z-\frac{1}{2}\right)-Z\left(z-\frac{i}{2}\right)\right)-i\pi+\frac{\pi^2(1+i)}{4e_1}\right).$$

Analog,

$$C_2(z)=\int_{(1+i)/2}^{z}\frac{i}{2}f(w)\left(1+g(w)^2\right)dw=\frac{i}{2}\left(-Z(w)-\pi w-\frac{\pi}{2e_1}\left(Z\left(w-\frac{1}{2}\right)-Z\left(w-\frac{i}{2}\right)\right)\right)\Bigg|_{(1+i)/2}^{z}=$$

$$=\frac{i}{2}\left(-Z(z)-\pi z-\frac{\pi}{2e_1}\left(Z\left(z-\frac{1}{2}\right)-Z\left(z-\frac{i}{2}\right)\right)+\pi-\frac{\pi^2(1+i)}{4e_1}\right).$$

Efectuam calculele si pentru ultima componenta a curbei minimale:

$f(w)g(w)=\dfrac{AP(w)}{P'(w)}=\dfrac{AP'(w)P(w)}{P'(w)^2}$. Cum $\wp'(z)^2=4\wp(z)^4-g_2\wp(z)-g_3$, rezulta ca

$$f(w)g(w)=\frac{AP(w)}{A\left(P'(w)^2-e_1^2\right)}=\frac{A}{8e_1}P'(w)\left(\frac{1}{P(w)-e_1}-\frac{1}{P(w)+e_1}\right)=\frac{\sqrt{2\pi}}{2}\left(\frac{P'(w)}{P(w)-e_1}-\frac{P'(w)}{P(w)+e_1}\right).$$

$$C_3(z)=\int_{(1+i)/2}^{z}f(w)g(w)dw=\frac{\sqrt{2\pi}}{4}\ln\left(\frac{P(w)-e_1}{P(w)+e_1}\right)\Bigg|_{(1+i)/2}^{z}=\frac{\sqrt{2\pi}}{4}\left(\ln\left(\frac{P(w)-e_1}{P(w)+e_1}\right)-\pi i\right).$$

Ca o consecinta directa a teoremei anterioare si a *Lemei IV.13.* avem urmatorul rezultat care ne da o definitie a suprafetei minimale Costa.

Corolar V.8.

Suprafata minimala Costa este data de *Costa(u,v) = (Costa$_1$(u,v), Costa$_2$(u,v), Costa$_3$(u,v))* unde

$$\begin{cases} Costa_1(z) = \dfrac{1}{2}\ \mathcal{R}e\left(-Z(u+iv)+\pi u+\dfrac{\pi^2}{4e_1}+\dfrac{\pi}{2e_1}\left(Z\left(u+iv-\dfrac{1}{2}\right)-Z\left(u+iv-\dfrac{i}{2}\right)\right)\right) \\ Costa_2(z) = \dfrac{1}{2}\ \mathcal{R}e\left(-iZ(u+iv)+\pi v+\dfrac{\pi^2}{4e_1}-\dfrac{\pi i}{2e_1}\left(Z\left(u+iv-\dfrac{1}{2}\right)-Z\left(u+iv-\dfrac{i}{2}\right)\right)\right) \\ Costa_3(z) = \dfrac{\sqrt{2\pi}}{4}\ln\left|\dfrac{P(u+iv)-e_1}{P(u+iv)+e_1}\right| \end{cases}$$

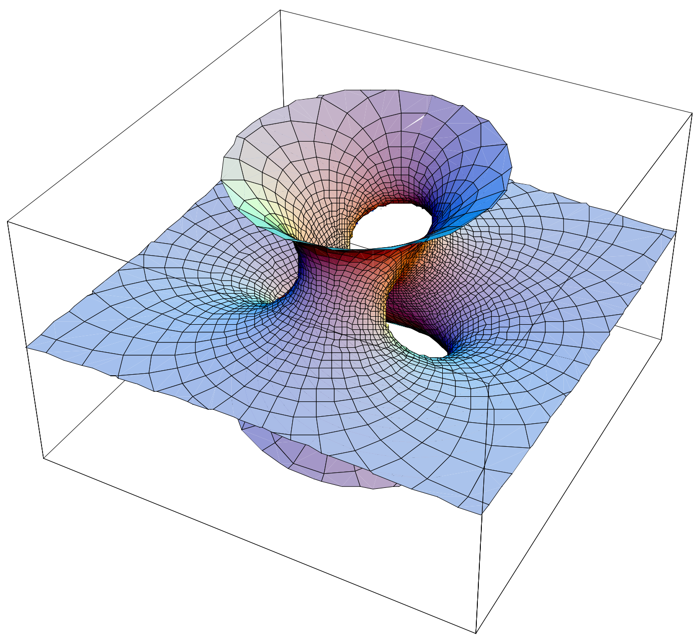


Fig. IV.10. Suprafata Costa

A N E X A   1

1. Operatorul Weingarten. Curburile principale

Fie $\mathcal{M}$ o suprafata regulata din $\mathbb{R}^3$, N normala Gauss, E, F, G coeficientii primei forme fundamentale si e, f, g coeficientii celei de-a doua forme fundamentale.

Teorema 1.1.

Fie $x(u,v)$ parametrizare a suprafetei $\mathcal{M}$ in punctul p. Atunci curbura Gauss si curbura medie sunt date de formulele $K = \dfrac{eg - f^2}{EG - F^2}$, $H = \dfrac{eG - 2fF + gE}{2(EG - F^2)}$, iar ecuatiile Weingarten se scriu sub forma $\begin{cases} Lx_u = \dfrac{eG - fF}{EG - F^2} x_u + \dfrac{fE - eF}{EG - F^2} x_v \\ Lx_v = \dfrac{fG - gF}{EG - F^2} x_u + \dfrac{gE - fF}{EG - F^2} x_v \end{cases}$.

Demonstratie. Vom incerca sa exprimam matricea operatorului Weingarten in functie de coeficientii celor doua forme fundamentale. Coeficientii celei de-a doua forme fundamentale sunt dati de expresiile

$e = - \langle N_u, x_u \rangle = \langle N, x_{uu} \rangle$

$f = - \langle N_v, x_u \rangle = \langle N, x_{uv} \rangle = \langle N, x_{vu} \rangle = - \langle N_u, x_v \rangle$

$g = - \langle N_v, x_v \rangle = \langle N, x_{vv} \rangle$,

unde $N_u = - L_{11}x_u - L_{21}x_v$, $N_v = - L_{12}x_u - L_{22}x_v$. Inlocuind, obtinem

$e = - \langle N_u, x_u \rangle = \langle L_{11}x_u + L_{21}x_v , x_u \rangle = L_{11}\langle x_u, x_u \rangle + L_{21}\langle x_v , x_u \rangle = L_{11}E + L_{21}F,$

$f = L_{11}F + L_{21}G,$

$f = L_{12}E + L_{22}F,$

$g = L_{12}F + L_{22}G,$

relatii ce pot fi scrise sub forma matriceala:

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \text{ de unde } \begin{pmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}, \text{ cu}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}, \text{ rezultand } \begin{pmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

Din utima egalitate obtinem matricea operatorului Weingarten:

$$L_{11} = \frac{eG - fF}{EG - F^2}, L_{12} = \frac{fG - gF}{EG - F^2}, L_{21} = \frac{fE - eF}{EG - F^2}, L_{22} = \frac{gE - fF}{EG - F^2},$$

$$L = \frac{1}{EG - F^2} \begin{pmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{pmatrix} .$$

Din definitie operatorului Weingarten si calculele efectuate anterior obtinem ecuatiile Weingarten:

$$\begin{cases} Lx_u = -N_u = L_{11}x_u + L_{21}x_v = \dfrac{eG - fF}{EG - F^2}x_u + \dfrac{fE - eF}{EG - F^2}x_v \\ Lx_v = -N_v = L_{12}x_u + L_{22}x_v = \dfrac{fG - gF}{EG - F^2}x_u + \dfrac{gE - fF}{EG - F^2}x_v. \end{cases}$$

Cum dN este dat de matricea operatorului Weingarten L, $K=\det L=\dfrac{eg - f^2}{EG - F^2}$, $H=\mathrm{Tr}L=\dfrac{eG - 2fF + gE}{2(EG - F^2)}$.

Cunoastem ca $K = k_1 k_2$ si $2H = k_1 + k_2$, de unde rezulta ca k_1 si k_2 sunt solutiile ecuatiei k^2-$2H$+K=0. Deci, $k_1 = H + \sqrt{H^2 - K}$ si $k_2 = H - \sqrt{H^2 - K}$.

Lema 1.2.
Fie $x{:}U \to \mathbb{R}^3$ parametrizarea suprafetei $\mathcal{M}$. Atunci coeficientii celei de-a doua forme fundamentale se pot scrie sub forma

$$\begin{cases} e = \dfrac{(x_{uu}, x_u, x_v)}{\sqrt{EG - F^2}} = \dfrac{\det\begin{pmatrix} x_{uu} \\ x_u \\ x_v \end{pmatrix}}{\sqrt{EG - F^2}} \\ f = \dfrac{(x_{uv}, x_u, x_v)}{\sqrt{EG - F^2}} = \dfrac{\det\begin{pmatrix} x_{uv} \\ x_u \\ x_v \end{pmatrix}}{\sqrt{EG - F^2}} \\ g = \dfrac{(x_{vv}, x_u, x_v)}{\sqrt{EG - F^2}} = \dfrac{\det\begin{pmatrix} x_v \\ x_u \\ x_v \end{pmatrix}}{\sqrt{EG - F^2}} \end{cases}$$

Demonstratie. Din definitie avem $e = -<N_u, x_u> = <N, x_{u\,u}> = < x_{uu}, \dfrac{x_u \times x_v}{\|x_u \times x_v\|} > = \dfrac{(x_{uu}, x_u, x_v)}{\|x_u \times x_v\|}$. In plus, avem in general, relatia $\|u\|^2\|v\|^2 = \|u \times v\|^2 + < u, v >^2$, de unde $\|u \times v\|^2 = \|u\|^2\|v\|^2 - < u, v >^2$, $\|u \times v\| = \sqrt{< u, u >< v, v > - < u, v >^2}$.

Atunci $\|x_u \times x_v\| = \sqrt{< x_u, x_u >< x_v, x_v > - < x_u, x_v >^2} = \sqrt{EG - F^2}$, deci $e = \dfrac{(x_{uu}, x_u, x_v)}{\sqrt{EG - F^2}}$.

Analog se obtin si celelalte doua scrieri, pentru f, respectiv g.

Corolar 1.3.
Fie $x\colon U\to\mathbb{R}^3$ parametrizarea suprafetei $\mathcal{M}$. Atunci curbura Gauss si cea medie se pot scrie sub forma

$$K=\frac{(x_{uu},x_u,x_v)(x_{vv},x_u,x_v)-(x_{uv},x_u,x_v)^2}{\left(\|x_u\|^2\|x_v\|^2-<x_u,x_v>^2\right)^2}$$

$$H=\frac{(x_{uu},x_u,x_v)\|x_v\|^2-2(x_{uv},x_u,x_v)<x_u,x_v>+(x_{vv},x_u,x_v)\|x_u\|^2}{2\left(\|x_u\|^2\|x_v\|^2-<x_u,x_v>^2\right)^{\frac{3}{2}}}$$

2. Curbura Gauss

Urmatoarele rezultate reprezinta, practic, formule de calcul ale curburii Gauss a unei suprafete in termeni de produse scalare ale derivatelor de ordinul intai si al doilea ale parametrizarii x a unei suprafete.

Teorema 2.1.
Fie $x\colon U\to\mathbb{R}^3$ parametrizarea suprafetei $\mathcal{M}$. Atunci curbura Gauss este data de

$$K=\frac{1}{(EG-F^2)^2}\left\{\det\begin{pmatrix}<x_{uu},x_{vv}> & <x_{uu},x_u> & <x_{uu},x_v>\\ <x_u,x_{vv}> & <x_u,x_u> & <x_u,x_v>\\ <x_v,x_{vv}> & <x_v,x_u> & <x_v,x_v>\end{pmatrix}-\det\begin{pmatrix}<x_{uv},x_{uv}> & <x_{uv},x_u> & <x_{uv},x_v>\\ <x_u,x_{uv}> & <x_u,x_u> & <x_u,x_v>\\ <x_v,x_{uv}> & <x_v,x_u> & <x_v,x_v>\end{pmatrix}\right\}$$

Demonstratie. Din *Corolarul 1.3* avem ca $K=\dfrac{(x_{uu},x_u,x_v)(x_{vv},x_u,x_v)-(x_{uv},x_u,x_v)^2}{\left(EG-F^2\right)^2}$ si scriind x = (x_1, x_2, x_3) obtinem

$$(x_{uu},x_u,x_v)(x_{vv},x_u,x_v)=\det\begin{pmatrix}x_{1uu} & x_{2uu} & x_{3uu}\\ x_{1u} & x_{2u} & x_{3u}\\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}\det\begin{pmatrix}x_{1vv} & x_{2vv} & x_{3vv}\\ x_{1u} & x_{2u} & x_{3u}\\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}=$$

$$=\det\left(\begin{pmatrix}x_{1uu} & x_{2uu} & x_{3uu}\\ x_{1u} & x_{2u} & x_{3u}\\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}\begin{pmatrix}x_{1vv} & x_{2vv} & x_{3vv}\\ x_{1u} & x_{2u} & x_{3u}\\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}\right)=\det\begin{pmatrix}<x_{uu},x_{vv}> & <x_{uu},x_u> & <x_{uu},x_v>\\ <x_u,x_{vv}> & <x_u,x_u> & <x_u,x_v>\\ <x_v,x_{vv}> & <x_u,x_v> & <x_v,x_v>\end{pmatrix}$$

$$(x_{uv},x_u,x_v)^2=\det\begin{pmatrix}x_{1vv} & x_{2vv} & x_{3vv}\\ x_{1u} & x_{2u} & x_{3u}\\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}\det\begin{pmatrix}x_{1vv} & x_{2vv} & x_{3vv}\\ x_{1u} & x_{2u} & x_{3u}\\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}$$

$$=\det\left(\begin{pmatrix} x_{1vv} & x_{2vv} & x_{3vv} \\ x_{1u} & x_{2u} & x_{3u} \\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}\begin{pmatrix} x_{1vv} & x_{2vv} & x_{3vv} \\ x_{1u} & x_{2u} & x_{3u} \\ x_{1v} & x_{2v} & x_{3v}\end{pmatrix}\right)=\det\begin{pmatrix} <x_{uv},x_{uv}> & <x_{uv},x_u> & <x_{uv},x_v> \\ <x_u,x_{uv}> & <x_u,x_u> & <x_u,x_v> \\ <x_v,x_{uv}> & <x_v,x_u> & <x_v,x_v>\end{pmatrix}$$

Lema 2.2.
Fie $x{:}U\to\mathbb{R}^3$ parametrizarea suprafetei $\mathcal{M}$. Atunci

$$<x_{uu}, x_{vv}> - < x_{uv}, x_{uv}> = -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} \ .$$

Demonstratie. $<x_u, x_{vv}>_u - < x_u, x_{uv}>_v = <x_{uu}, x_{vv}>+<x_u, x_{vvu}> - < x_{uv}, x_{uv}> - < x_u, x_{uvv}>$

Deci $\underline{<x_{uu}, x_{vv}> - < x_{uv}, x_{uv}> = <x_u, x_{vv}>_u - < x_u, x_{uv}>_v}$ **(1)**

Evaluam fiecare termen din membrul drept .

$(<x_u, x_v>)_v = <x_{uv}, x_v>+<x_u, x_{vv}> \Rightarrow <x_u, x_{vv}> = (<x_u, x_v>)_v - <x_{uv}, x_v>$

$\Rightarrow (<x_u, x_{vv}>)_u = ((<x_u, x_v>)_v - <x_{uv}, x_v>)_u = (<x_u, x_v>)_{vu} - (<x_{uv}, x_v>)_u$

Dar $(<x_v, x_v>)_u = <x_{uv}, x_v> + <x_v, x_{vu}> = 2 <x_{uv}, x_v> \Rightarrow <x_{uv}, x_v> = \frac{1}{2}(<x_v, x_v>)_u$

$\Rightarrow \underline{(<x_u, x_{vv}>)_u = (<x_u, x_v>)_{vu} - \frac{1}{2}(<x_v, x_v>)_{uu}}$ **(2)**

$(<x_u, x_u>)_v =<x_{uv}, x_u>+<x_u, x_{vu}>=2<x_{uv}, x_u> \Rightarrow <x_{uv}, x_u> = \frac{1}{2}(<x_u, x_u>)_v$

$\Rightarrow \underline{(<x_{uv}, x_u>)_v = \frac{1}{2}(<x_u, x_u>)_{vv}}$ **(3)**

Deci $<x_{uu}, x_{vv}> - < x_{uv}, x_{uv}> = (<x_u, x_v>)_{vu} - \frac{1}{2}(<x_v, x_v>)_{uu} - \frac{1}{2}(<x_u, x_u>)_{vv}$, adica

$<x_{uu}, x_{vv}> - < x_{uv}, x_{uv}> = -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu}$ q.e.d.

Teorema 2.3. ***(Formula lui Brioschi)***
Fie $x{:}U\to\mathbb{R}^3$ parametrizarea suprafetei $\mathcal{M}$. Atunci curbura Gauss se poate scrie sub forma K

$$=\frac{1}{(EG-F^2)^2}\left\{\det\begin{pmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G\end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G\end{pmatrix}\right\}.$$

Demonstratie. Vom folosi scrierea curburii Gauss sub forma de diferenta de determinanti din *Teorema 2.1.* si vom calcula elementele care apar in scrierea celor doi determinanti, folosind si calcule efectuate in *Lema 2.2.*

$(<x_u,x_u>)_u = <x_{uu},x_u>+<x_u,x_{uu}> = 2<x_{uu},x_u> \Rightarrow <x_{uu},x_u> = \frac{1}{2}(<x_u,x_u>)_u \Rightarrow \underline{<x_{uu},x_u> = \frac{1}{2}E_u}$;

$(<x_u, x_v>)_u = <x_{uu}, x_v>+<x_u, x_{uv}> \Rightarrow <x_{uu}, x_v> = (<x_u, x_v>)_u - <x_u, x_{vu}>$

Dar $(<x_u, x_u>)_v = <x_{uv}, x_u> + <x_u, x_{vu}> = 2 <x_{uv}, x_u> \Rightarrow <x_{uv}, x_u> = \frac{1}{2}(<x_u, x_u>)_v$

$\Rightarrow <x_{uu}, x_v> = (<x_u, x_v>)_u - \frac{1}{2}(<x_u, x_u>)_v \quad \Rightarrow \underline{<x_{uu}, x_v> = F_u - \frac{1}{2}E_v}$;

Din *Lema 2.2*, relatia *(2)* avem ca

$<x_u, x_{vv}> = (<x_u, x_v>)_v - \frac{1}{2}(<x_v, x_v>)_u \Rightarrow \underline{<x_u, x_{vv}> = F_v - \frac{1}{2}G_u}$;

$(<x_v,x_v>)_v = <x_{vv},x_v>+<x_v,x_{vv}> = 2<x_{vv},x_v> \Rightarrow <x_{vv},x_v> = \frac{1}{2}(<x_v,x_v>)_v \Rightarrow \underline{<x_{vv},x_v> = \frac{1}{2}G_v}$;

$(<x_u,x_u>)_v = <x_{uv},x_u>+<x_u,x_{uv}> = 2<x_u,x_{uv}> \Rightarrow <x_u,x_{uv}> = \frac{1}{2}(<x_u,x_u>)_v \Rightarrow \underline{<x_u,x_{uv}> = \frac{1}{2}E_v}$;

$(<x_v,x_v>)_u = <x_{uv},x_v>+<x_v,x_{uv}> = 2<x_{uv},x_v> \Rightarrow <x_{uv},x_v> = \frac{1}{2}(<x_v,x_v>)_u \Rightarrow \underline{<x_{uv},x_v> = \frac{1}{2}G_u}$;

Asadar, folosind calculele anterioare si *Teorema 2.1.* avem urmatoarea scriere pentru curbura Gauss:

$$K = \frac{1}{(EG-F^2)^2}\left\{\det\begin{pmatrix} <x_{uu},x_{vv}> & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix} - \det\begin{pmatrix} <x_{uv},x_{uv}> & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}\right\}.$$

Observam ca, la un calcul al determinantilor prin dezvoltare dupa prima linie, $<x_{uu},x_{vv}>$ si $<x_{uv},x_{uv}>$ se vor inmulti cu determinantul aceleiasi submatrici $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$, deci acelasi coeficient, $EG - F^2$. In concluzie, putem rescrie curbura Gauss sub forma:

$$K = \frac{1}{(EG-F^2)^2}\left\{\det\begin{pmatrix} <x_{uu},x_{vv}> - <x_{uv},x_{uv}> & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}\right\}$$

si, din *Lema 2.2,* cunoastem ca $<x_{uu}, x_{vv}> - <x_{uv}, x_{uv}> = -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu}$, de unde rezulta formula Brioschi.

Corolar 2.4.
Fie $x{:}U\to\mathbb{R}^3$ parametrizarea suprafetei $\mathcal{M}$ pentru care $F = 0$. Atunci curbura Gauss este data de:

$$K=\frac{-1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right\}=\frac{-1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)+\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right\}.$$

Demonstratie. Pentru $F = 0$ formula lui Brioschi devine

$$K=\frac{1}{(EG)^2}\left\{\det\begin{pmatrix}-\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & -\frac{1}{2}E_v\\ -\frac{1}{2}G_u & E & 0\\ \frac{1}{2}G_v & 0 & G\end{pmatrix}-\det\begin{pmatrix}0 & \frac{1}{2}E_v & \frac{1}{2}G_u\\ \frac{1}{2}E_v & E & 0\\ \frac{1}{2}G_u & 0 & G\end{pmatrix}\right\}=$$

$$=\frac{1}{(EG)^2}\left\{\left(-\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu}\right)EG+\frac{1}{4}E_uG_uG+\frac{1}{4}E_vG_vE+\frac{1}{4}E_v^{\ 2}G+\frac{1}{4}G_u^{\ 2}E\right\}=$$

$$=-\frac{1}{2}\frac{E_{vv}}{EG}+\frac{1}{4}\frac{E_v^{\ 2}}{E^2G}+\frac{1}{4}\frac{E_vG_v}{EG^2}-\frac{1}{2}\frac{G_{uu}}{EG}+\frac{1}{4}\frac{G_u^{\ 2}}{EG^2}+\frac{1}{4}\frac{E_uG_u}{E^2G}\quad .$$

Pe de alta parte,

$$\frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)=\frac{1}{\sqrt{EG}}\left\{-\frac{G_v}{2G^{\frac{3}{2}}}\frac{\partial\sqrt{E}}{\partial v}+\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{E}}{\partial v^2}\right\}=\frac{1}{\sqrt{EG}}\left\{-\frac{G_vE_v}{4G^{\frac{3}{2}}\sqrt{E}}+\frac{1}{2\sqrt{G}}\frac{\partial}{\partial v}\left(E^{-\frac{1}{2}}E_v\right)\right\}$$

$$=\frac{1}{\sqrt{EG}}\left\{-\frac{G_vE_v}{4G^{\frac{3}{2}}\sqrt{E}}+\frac{1}{2\sqrt{G}}\left(-\frac{E_v^{\ 2}}{2E^{\frac{3}{2}}}+\frac{E_{vv}}{\sqrt{E}}\right)\right\}=-\frac{G_vE_v}{4G^2E}-\frac{E_v^{\ 2}}{4E^2G}+\frac{E_{vv}}{2EG}.$$

Similar,

$$\frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)=\frac{1}{\sqrt{EG}}\left\{-\frac{E_u}{2E^{\frac{3}{2}}}\frac{\partial\sqrt{G}}{\partial u}+\frac{1}{\sqrt{E}}\frac{\partial^2\sqrt{G}}{\partial u^2}\right\}=\frac{1}{\sqrt{EG}}\left\{-\frac{E_uG_u}{4E^{\frac{3}{2}}\sqrt{G}}+\frac{1}{2\sqrt{E}}\frac{\partial}{\partial u}\left(G^{-\frac{1}{2}}G_u\right)\right\}$$

$$=\frac{1}{\sqrt{EG}}\left\{-\frac{E_uG_u}{4E^{\frac{3}{2}}\sqrt{G}}+\frac{1}{2\sqrt{E}}\left(-\frac{G_u^{\ 2}}{2G^{\frac{3}{2}}}+\frac{G_{uu}}{\sqrt{G}}\right)\right\}=-\frac{E_uG_u}{4E^2G}-\frac{G_u^{\ 2}}{4G^2E}+\frac{G_{uu}}{2EG}.$$

Prin insumarea ultimilor doua expresii luate cu semn schimbat obtinem scrierea curburii Gauss K rezultata din calculul diferentei de determinanti.

A N E X A   2

1. Curburile unei suprafete de rotatie

Fie $\mathcal{M}$ suprafata de rotatie parametrizata prin $x:(0,2\pi)\times(a,b)\to\mathbb{R}^3$, $x(u,v)=(\varphi(v)\cos u,\varphi(v)\sin u,\psi(v))$.

Calculam coeficientii primei forme fundamentale.

$x_u = (-\varphi(v)\sin u,\ \varphi(v)\cos u,\ 0)$, $\qquad x_v = (\varphi'(v)\cos u,\ \varphi'(v)\sin u,\ \psi'(v))$,

$x_{uu} = (-\varphi(v)\cos u,\ -\varphi(v)\sin u,\ 0)$, $\qquad x_{uv} = x_{vu} = (-\varphi'(v)\sin u,\ \varphi'(v)\cos u,\ 0)$,

$x_{vv} = (\varphi''(v)\cos u,\ \varphi''(v)\sin u,\ \psi''(v))$.

$E = \langle x_u,x_u\rangle = \varphi^2\sin^2 u+\varphi^2\cos^2 u = \varphi^2$,

$F = \langle x_u,x_v\rangle = \langle x_v,x_u\rangle = -\varphi\varphi'\sin u\cos u + \varphi\varphi'\cos u\sin u = 0$,

$G = \langle x_v,x_v\rangle = (\varphi')^2\cos^2 u + (\varphi')^2\sin^2 u + (\psi')^2 = (\varphi')^2 + (\psi')^2$.

Calculam coeficientii celei de-a doua forme fundamentale.

Normala Gauss este $N = \dfrac{x_u\times x_v}{\|x_u\times x_v\|}$,

$$x_u\times x_v = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\varphi\sin u & \varphi\cos u & 0 \\ \varphi'\cos u & \varphi'\sin u & \psi' \end{vmatrix} = (\varphi\psi'\cos u,\ -\varphi\psi'\sin u,\ -\varphi\varphi')$$

$$\|x_u\times x_v\| = \sqrt{\varphi^2(\psi')^2+\varphi^2(\varphi')^2} = \varphi\sqrt{(\psi')^2+(\varphi')^2}\,.$$

$$e = \langle N,x_{uu}\rangle = \left\langle \frac{x_u\times x_v}{\|x_u\times x_v\|}, x_{uu}\right\rangle = \frac{(x_u,x_v,x_{uu})}{\|x_u\times x_v\|}$$

$$=\frac{1}{\varphi\sqrt{(\varphi')^2+(\psi')^2}}\begin{vmatrix} -\varphi\sin u & \varphi\cos u & 0 \\ \varphi'\cos u & \varphi'\sin u & \psi' \\ -\varphi\cos u & -\varphi\sin u & 0 \end{vmatrix} = \frac{-\psi'\varphi^2(\sin^2 u+\cos^2 u)}{\varphi\sqrt{(\varphi')^2+(\psi')^2}} = \frac{-\psi'\varphi}{\sqrt{(\varphi')^2+(\psi')^2}}\,.$$

$$f = \langle N,x_{uv}\rangle = \left\langle \frac{x_u\times x_v}{\|x_u\times x_v\|}, x_{uv}\right\rangle = \frac{(x_u,x_v,x_{uv})}{\|x_u\times x_v\|} =$$

$$=\frac{1}{\varphi\sqrt{(\varphi')^2+(\psi')^2}}\begin{vmatrix} -\varphi\sin u & \varphi\cos u & 0 \\ \varphi'\cos u & \varphi'\sin u & \psi' \\ -\varphi'\sin u & \varphi'\cos u & 0 \end{vmatrix} = \frac{-\varphi\varphi'\sin u\cos u+\varphi\varphi'\sin u\cos u}{\varphi\sqrt{(\varphi')^2+(\psi')^2}} = 0.$$

$$g = \langle N,x_{vv}\rangle = \left\langle \frac{x_u\times x_v}{\|x_u\times x_v\|}, x_{vv}\right\rangle = \frac{(x_u,x_v,x_{vv})}{\|x_u\times x_v\|} =$$

$$=\frac{1}{\varphi\sqrt{(\varphi')^2+(\psi')^2}}\begin{vmatrix}-\varphi\sin u & \varphi\cos u & 0\\ \varphi'\cos u & \varphi'\sin u & \psi'\\ \varphi''\cos u & \varphi''\sin u & \psi''\end{vmatrix}=$$

$$=\frac{\psi'\varphi\varphi''(\sin^2 u+\cos^2 u)-\psi''\varphi'\varphi(\sin^2 u+\cos^2 u)}{\varphi\sqrt{(\varphi')^2+(\psi')^2}}=$$

$$=\frac{\psi'\varphi''-\psi''\varphi}{\sqrt{(\varphi')^2+(\psi')^2}}'.$$

Folosind *Teorema 1.1.* din *Anexa 1* si calculele efectuate anterior obtinem

$$K=\frac{eg-f^2}{EG-F^2}=\frac{eg}{EG}=\frac{-\psi'\varphi(\psi'\varphi''-\psi''\varphi')}{\varphi^2((\varphi')^2+(\psi')^2)}=\frac{-\psi'(\psi'\varphi''-\psi''\varphi')}{\varphi((\varphi')^2+(\psi')^2)},$$

$$H=\frac{eG-2fF+gE}{2(EG-F^2)}=\frac{eG+gE}{2EG}=\frac{e}{2E}+\frac{g}{2G}=\frac{-\psi'\varphi}{2\varphi^2}+\frac{\psi'\varphi''-\psi''\varphi'}{2((\varphi')^2+(\psi')^2)}=$$

$$=\frac{1}{2}\,\frac{-\psi'(((\varphi')^2+(\psi')^2)+\varphi(\psi'\varphi''-\psi''\varphi')}{\varphi((\varphi')^2+(\psi')^2)},$$

$$k_1=H+\sqrt{H^2-K}=\frac{eG+gE}{2EG}+\sqrt{\left(\frac{eG+gE}{2EG}\right)^2-\frac{eg}{EG}}=\frac{e}{E}=\frac{-\psi'}{\varphi\sqrt{(\varphi')^2+(\psi')^2}},$$

$$k_2=H-\sqrt{H^2-K}=\frac{eG+gE}{2EG}-\sqrt{\left(\frac{eG+gE}{2EG}\right)^2-\frac{eg}{EG}}=\frac{g}{G}=\frac{\psi'\varphi''-\psi''\varphi}{((\varphi')^2+(\psi')^2)^{3/2}}'.$$

Observatie:

Daca profilul (curba generatoare) suprafetei $\mathcal{M}$, $c(t)=(\varphi(t),0,\psi(t))$, este parametrizata canonic, atunci $(\varphi')^2+(\psi')^2=1$ si formulele anterioare devin:

$$K=\frac{-\psi'(\psi'\varphi''-\psi''\varphi')}{\varphi},$$

$$H=\frac{1}{2}\,\frac{-\psi'+\varphi(\psi'\varphi''-\psi''\varphi')}{\varphi},$$

$$k_1=\frac{-\psi'}{\varphi},$$

$$k_2=\psi'\varphi''-\psi''\varphi'.$$

2. Curburile unei parametrizari Monge

Fie $h\colon U \to \mathbb{R}$ diferentiabila, $U \subset \mathbb{R}^2$ deschisa si suprafata $\mathcal{M}$ data de $x(u,v) = (u, v, h(u,v))$, unde $x\colon U \to \mathcal{M}$ (parametrizare Monge).

Calculam coeficientii primei forme fundamentale.

$x_u = (1,0,h_u)$, $x_v = (0,1,h_v)$, $x_{uu} = (0,0,h_{uu})$, $x_{uv} = x_{vu} = (0,0,h_{uv})$, $x_{vv} = (0,0,h_{vv})$.

$E = \langle x_u, x_u \rangle = 1 + h_u^2$,

$F = \langle x_u, x_v \rangle = \langle x_v, x_u \rangle = h_u h_v$,

$G = \langle x_v, x_v \rangle = 1 + h_v^2$.

Calculam coeficientii celei de-a doua forme fundamentale.

Normala Gauss este $N = \dfrac{x_u \times x_v}{\|x_u \times x_v\|}$,

$$x_u \times x_v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & h_u \\ 0 & 1 & h_v \end{vmatrix} = (-h_u, h_v, 1), \qquad \|x_u \times x_v\| = \sqrt{1 + h_u^2 + h_v^2} .$$

$$e = \frac{(x_u, x_v, x_{uu})}{\|x_u \times x_v\|} = \frac{1}{\sqrt{1+h_u^2+h_v^2}} \begin{vmatrix} 1 & 0 & h_u \\ 0 & 1 & h_v \\ 0 & 0 & h_{uu} \end{vmatrix} = \frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}} .$$

$$f = \frac{(x_u, x_v, x_{uv})}{\|x_u \times x_v\|} = \frac{1}{\sqrt{1+h_u^2+h_v^2}} \begin{vmatrix} 1 & 0 & h_u \\ 0 & 1 & h_v \\ 0 & 0 & h_{uv} \end{vmatrix} = \frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}} .$$

$$g = \frac{(x_u, x_v, xvv)}{\|x_u \times x_v\|} = \frac{1}{\sqrt{1+h_u^2+h_v^2}} \begin{vmatrix} 1 & 0 & h_u \\ 0 & 1 & h_v \\ 0 & 0 & h_{vv} \end{vmatrix} = \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} .$$

Din *Teorema 1.1.* din *Anexa 1* si calculele anterioare putem calcula curbura Gauss, curbura normala si curburile principale:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2} ,$$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{1}{2}\,\frac{h_{uu}(1+h_v^2) - 2h_{uv}h_u h_v + h_{vv}(1+h_u^2)}{(1+h_u^2+h_v^2)^{3/2}} ,$$

$$k_1 = \frac{e}{E} = \frac{h_{uu}}{(1+h_u^2)\sqrt{1+h_u^2+h_v^2}} , \qquad k_2 = \frac{g}{G} = \frac{h_{vv}}{(1+h_v^2)\sqrt{1+h_u^2+h_v^2}} .$$

A N E X A 3

1. Suprafete minimale izoterme conjugate armonic

Propozitia 1.1.
Elicoidul si catenoidul sunt suprafete minimale izoterme conjugate armonic.
Demonstratie. Fie elicoidul dat de parametrizarea $x(u,v) = (b\ sh\ v\ cos u,\ b\ sh\ v\ sin u,\ bu)$ si catenoidul dat de $y(u,v)=\left(a\cos u\,ch\left(\frac{v}{a}\right), a\sin u\,ch\left(\frac{v}{a}\right), v\right)$. Pentru simplitate vom considera $a = b = 1$.
Vom arata mai intai ca cele doua suprafete sunt izoterme verificand definitia: $<x_u\ ,\ x_u> = <x_v\ ,\ x_v> = \lambda^2$ si $<x_u\ ,x_v> = 0$.
Catenoidul este suprafata de rotatie si, folosind rezultatele din *Anexa 2.1.* ($\varphi(v) = ch\ v$ si $\psi(u)=v$) obtinem:

$y_u = (\ -\ ch\ v\ sin\ u,\ ch\ v\ cos\ u,\ 0)\ ,\quad y_v = (\ sh\ v\ cos\ u,\ sh\ v\ sin\ u,\ 1)\ ;$
$E\ = <y_u,y_u> = ch^2\ v\ ;$
$G\ = <y_v\ ,y_v> = ((ch\ v)')^2+1= sh^2\ v + 1 = ch^2 v$, deci $<y_u\ ,y_u> = <y_v\ ,y_v>= \lambda^2$, cu $\lambda^2 = ch^2 v$;
$F\ = <y_u,y_v> = 0$.

Deci catenoidul este suprafata izoterma.
Efectuam acum calculele pentru elicoid:

$x_u = (\ -\ sh\ v\ sin\ u,\ sh\ v\ cos\ u,\ 1)\ ,\quad x_v = (\ ch\ v\ cos\ u,\ ch\ v\ sin\ u,\ 0)\ ;$
$E\ = <x_u,x_u> = sh^2\ v\ sin^2\ u + sh2\ v\ cos^2\ u + 1 = sh^2\ v + 1 = ch^2\ v\ ;$
$G\ = <x_v\ ,x_v> = ch^2\ v\ cos^2\ u + ch^2\ v\ sin^2\ u = ch^2\ v$ deci $<x_u\ ,x_u> = <x_v\ ,x_v>= \lambda^2$, cu $\lambda = ch\ v$;
$F\ = <x_u,x_v> = -\ sh\ v\ sin\ u\ ch\ v\ cos\ u + sh\ v\ cos\ u\ ch\ v\ sin\ u = 0.$

Deci si elicoidul este suprafata izoterma si ambele suprafete au aceeasi coeficienti ai primei forme fundamentale.
Se verifica si faptul ca sunt suprafete conjugate armonic, verifica relatiile Cauchy-Riemann.
Atunci, din *Teorema IV.7.* rezulta ca avem o deformare izometrica cu $z(t)$ suprafata izoterma minimala (deci toate suprafetele intermediare din procesul deformarii sunt minimale).

2. Functii complexe

Fie $z\in\mathbb{C}$, $z=u+iv$. Atunci, $\forall z\in\mathbb{C}$ $\quad e^{z} = e^{u+iv} = e^{u}\,(cos\ v + i\ sin\ v)$ si

$$cos\ z = \frac{e^{iz}+e^{-iz}}{2}\ ,\quad sin\ z = \frac{e^{iz}-e^{-iz}}{2i}\ ,\quad ch\ z = \frac{e^{z}+e^{-z}}{2}\ ,\quad sh\ z = \frac{e^{z}-e^{-z}}{2}\ ,$$

$$sin\ z = \frac{e^{iu-v}-e^{-iu+v}}{2i} = \frac{e^{-v}(\cos u + i\sin u) - e^{v}(\cos u - i\sin u)}{2i} = \frac{\sin u(e^{-v}-e^{v})}{2} + \frac{\cos u(e^{-v}-e^{v})}{2i} =$$

$= ch\ v\ sin\ u + i\ sh\ v\ cos\ u$,

$$cos\ z = \frac{e^{iu-v}+e^{-iu+v}}{2} = \frac{e^{-v}(\cos u + i\sin u)+e^{v}(\cos u - i\sin u)}{2} = \frac{\cos u(e^{-v}+e^{v})}{2} + i\frac{\sin u(e^{-v}-e^{v})}{2} =$$

$= ch\ v\ cos\ u\ + i\ sh\ v\ sin\ u$,

$$chz = ch(-iz) = \frac{e^{-iz}+e^{iz}}{2} = \cos(iz) \ ,$$

$$chz = \frac{e^{u+iv}+e^{-u-iv}}{2} = \frac{e^{u}(\cos v + i\sin v)+e^{-u}(\cos v - i\sin v)}{2} = \frac{\cos v(e^{u}+e^{-u})}{2} + i\frac{\sin v(e^{u}-e^{-u})}{2} =$$

$$= \frac{\cos v chu}{2} + i\frac{\sin v shu}{2} \ ,$$

$$shz = \frac{e^{u+iv}-e^{-u-iv}}{2} = \frac{e^{u}(\cos v + i\sin v)-e^{-u}(\cos v - i\sin v)}{2} = \frac{\cos v(e^{u}-e^{-u})}{2} + i\frac{\sin v(e^{u}+e^{-u})}{2} =$$

$$= \frac{\cos v shu}{2} + i\frac{\sin v chu}{2} \ ,$$

$$sin(iz) = \frac{e^{-z}-e^{z}}{2i} = i\ sh\ z \ , \quad sh(-iz) = \frac{e^{z}-e^{-z}}{2i} = -\ i\ sh\ z \ ,$$

$$shzchz = \frac{e^{u+iv}-e^{-u-iv}}{2}\frac{e^{u+iv}+e^{-u-iv}}{2} = \frac{e^{2(u+iv)}-e^{2(-u-iv)}}{4} =$$

$$= \frac{e^{2u}(\cos(2v)+i\sin(2v))-e^{-2u}(\cos(2v)-i\sin(2v))}{4} =$$

$$= \frac{\cos(2v)(e^{2u}-e^{-2u})}{4} + i\frac{\sin(2v)(e^{2u}+e^{-2u})}{4} = \frac{\cos(2v)sh(2u)}{2} + i\frac{\sin(2v)ch(2u)}{2}$$

$$sh^2 z = \left(\frac{e^{u+iv}-e^{-u-iv}}{2}\right)^2 = \frac{e^{2(u+iv)}+e^{2(-u-iv)}-2}{4} =$$

$$= -\frac{1}{2} + \frac{e^{2u}(\cos(2v)+i\sin(2v))+e^{-2u}(\cos(2v)-i\sin(2v))}{4} =$$

$$= -\frac{1}{2} + \frac{\cos(2v)(e^{2u}+e^{-2u})}{4} + i\frac{\sin(2v)(e^{2u}-e^{-2u})}{4} = -\frac{1}{2} + \frac{\cos(2v)ch(2u)}{2} + i\frac{\sin(2v)sh(2u)}{2}$$

$$ch^2 z = \left(\frac{e^{u+iv}+e^{-u-iv}}{2}\right)^2 = \frac{e^{2(u+iv)}+e^{2(-u-iv)}+2}{4} =$$

$$= \frac{1}{2} + \frac{e^{2u}(\cos(2v)+i\sin(2v))+e^{-2u}(\cos(2v)-i\sin(2v))}{4} =$$

$$= \frac{1}{2} + \frac{\cos(2v)(e^{2u}+e^{-2u})}{4} + i\frac{\sin(2v)(e^{2u}-e^{-2u})}{4} = \frac{1}{2} + \frac{\cos(2v)ch(2u)}{2} + i\frac{\sin(2v)sh(2u)}{2}$$

A N E X A 4

Vectori complecsi

Lema 1.1.
Fie $a,b \in \mathbb{C}^3$. Atunci avem egalitatile:

(i) $\|a\times\overline{b}\|^2 = \|a\|^2\|b\|^2 - |< a,\overline{b} >|^2$

(ii) $\|a\times\overline{a}\| = \|a\|^2$

(iii) $a\times\overline{a} = 2i(\mathfrak{Im}(a_2\overline{a_3}),\mathfrak{Im}(a_3\overline{a_1}),\mathfrak{Im}(a_1\overline{a_2}))$, unde $a = (a_1, a_2, a_3)$.

Demonstratie.

(i) $\|a\|^2\|b\|^2 = \|a\|^2\|\overline{b}\|^2 = \|a\|^2\|\overline{b}\|^2(\sin^2\varphi+\cos^2\varphi) = \|a\times\overline{b}\|^2 + |< a,\overline{b} >|^2$, $\varphi = \cos(a,\overline{b})$

(ii) $\|a\times\overline{a}\|^2 = < a\times\overline{a},\overline{a\times\overline{a}} > = - < a\times\overline{a},a\times\overline{a} > = - < a\times a,\overline{a}\times\overline{a} > + < a,\overline{a} >^2 = \|a\|^4$

$$(iii)\ a\times\overline{a} = \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \overline{a_1} & \overline{a_2} & \overline{a_3} \end{pmatrix} = (a_2\overline{a_3}-a_3\overline{a_2}, a_1\overline{a_3}-a_3\overline{a_1}, a_1\overline{a_2}-a_2\overline{a_1}).$$

Cum $a_1, a_2, a_3 \in \mathbb{C}$, vom nota $a_1 = a_{11} + i\,a_{12}$, $a_2 = a_{21} + i\,a_{22}$, $a_3 = a_{31} + i\,a_{32}$ si obtinem:

$a_2\overline{a_3}-a_3\overline{a_2} = (a_{21} + i\,a_{22})(a_{31} - i\,a_{32}) - (a_{31} + i\,a_{32})(a_{21} - i\,a_{22})$
$= 2i\,(a_{22}\,a_{31} - a_{21}\,a_{32})$.

$2i\,\mathfrak{Im}(a_2\overline{a_3}) = 2i\,\mathfrak{Im}[(a_{21} + i\,a_{22})(a_{31} - i\,a_{32})] = 2i\,\mathfrak{Im}(a_{21}\,a_{31} + a_{22}\,a_{32} + i(a_{31}\,a_{22} - a_{21}\,a_{32}))$
$= 2i(a_{31}\,a_{22} - a_{21}\,a_{32})$.

$\Rightarrow$ $a_2\overline{a_3}-a_3\overline{a_2} = 2i\,\mathfrak{Im}(a_2\overline{a_3})$. Analog se arata si pentru celelalte:

$a_1\overline{a_3}-a_3\overline{a_1} = (a_{11} + i\,a_{12})(a_{31} - i\,a_{32}) - (a_{31} + i\,a_{32})(a_{11} - i\,a_{12})$
$= 2i\,(a_{32}\,a_{11} - a_{31}\,a_{12})$.

$2i\,\mathfrak{Im}(a_3\overline{a_1}) = 2i\,\mathfrak{Im}[(a_{31} + i\,a_{32})(a_{11} - i\,a_{12})] = 2i\,\mathfrak{Im}(a_{31}\,a_{11} + a_{32}\,a_{12} + i(a_{32}\,a_{11} - a_{31}\,a_{12}))$
$= 2i(a_{32}\,a_{11} - a_{31}\,a_{12})$.

$\Rightarrow$ $a_1\overline{a_3}-a_3\overline{a_1} = 2i\,\mathfrak{Im}(a_3\overline{a_1})$;

$a_1\overline{a_2}-a_2\overline{a_1} = (a_{11} + i\,a_{11})(a_{21} - i\,a_{22}) - (a_{21} + i\,a_{22})(a_{11} - i\,a_{11})$
$= 2i\,(a_{32}\,a_{11} - a_{31}\,a_{12})$.

$2i\,\mathfrak{Im}(a_1\overline{a_2}) = 2i\,\mathfrak{Im}[(a_{11} + i\,a_{12})(a_{21} - i\,a_{22})] = 2i\,\mathfrak{Im}(a_{11}\,a_{12} + a_{12}\,a_{22} + i(a_{12}\,a_{21} - a_{11}\,a_{22}))$
$= 2i(a_{12}\,a_{21} - a_{11}\,a_{22})$.

$\Rightarrow$ $a_1\overline{a_2}-a_2\overline{a_1} = 2i\,\mathfrak{Im}(a_1\overline{a_2})$.

Anexa 5
Suprafete minimale

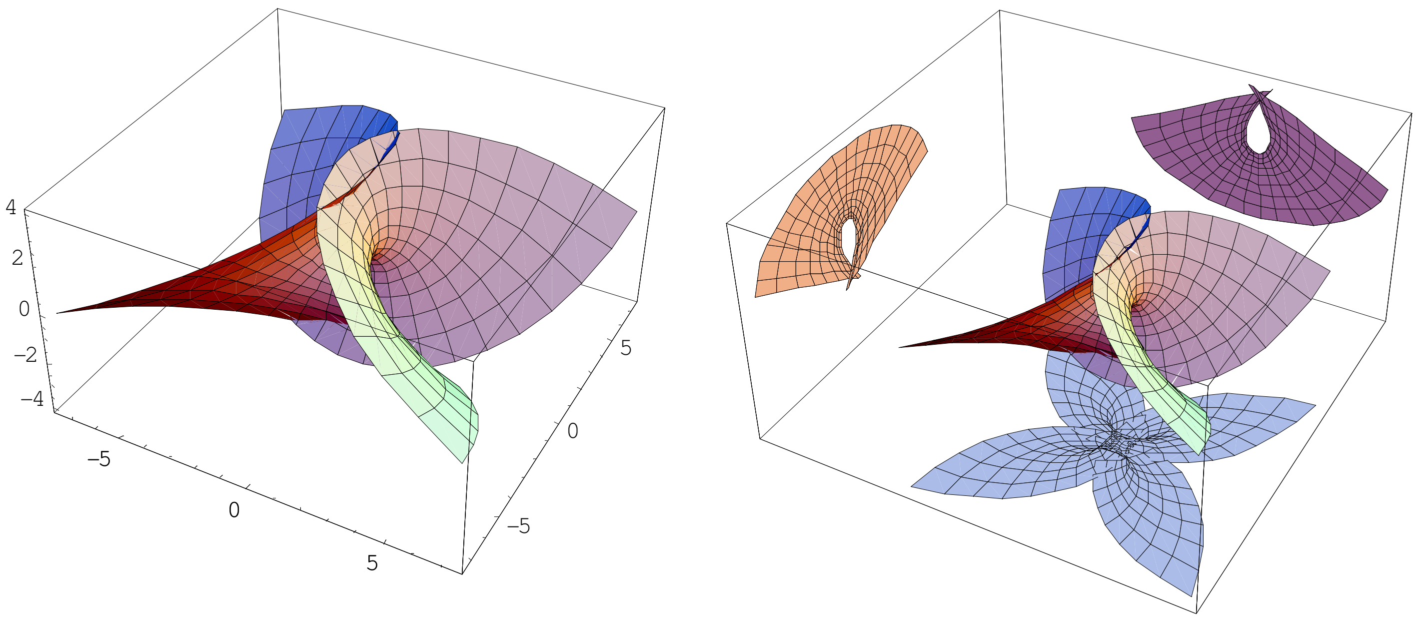


Fig.1.1. Suprafata Enneper

$$x(u,v) = \left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - vu^2, u^2 - v^2 \right)$$

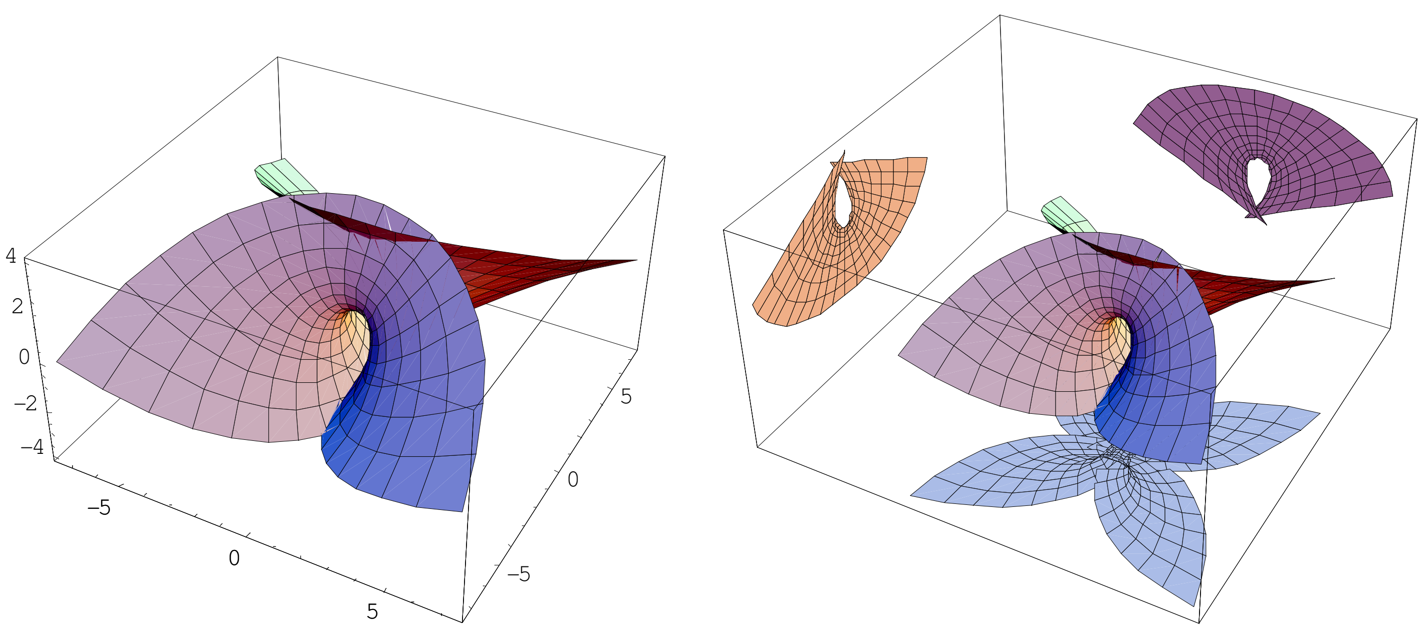


Fig.1.2. Suprafata Enneper

rotatie cu $\frac{\pi}{2}$ sau $-\frac{\pi}{2}$ in jurul axei Oz

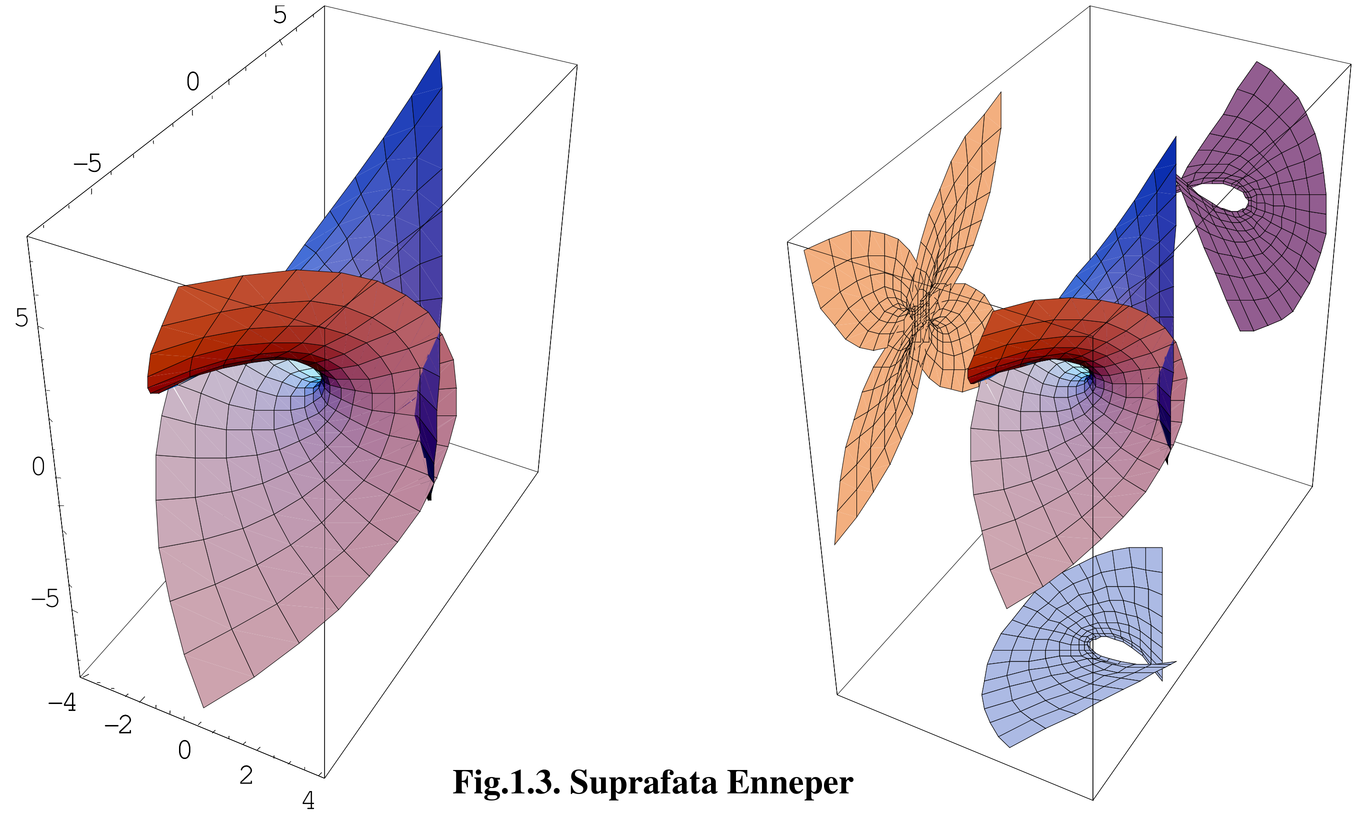


Fig.1.3. Suprafata Enneper

rotatie cu $-\dfrac{\pi}{2}$ in jurul axei Oy

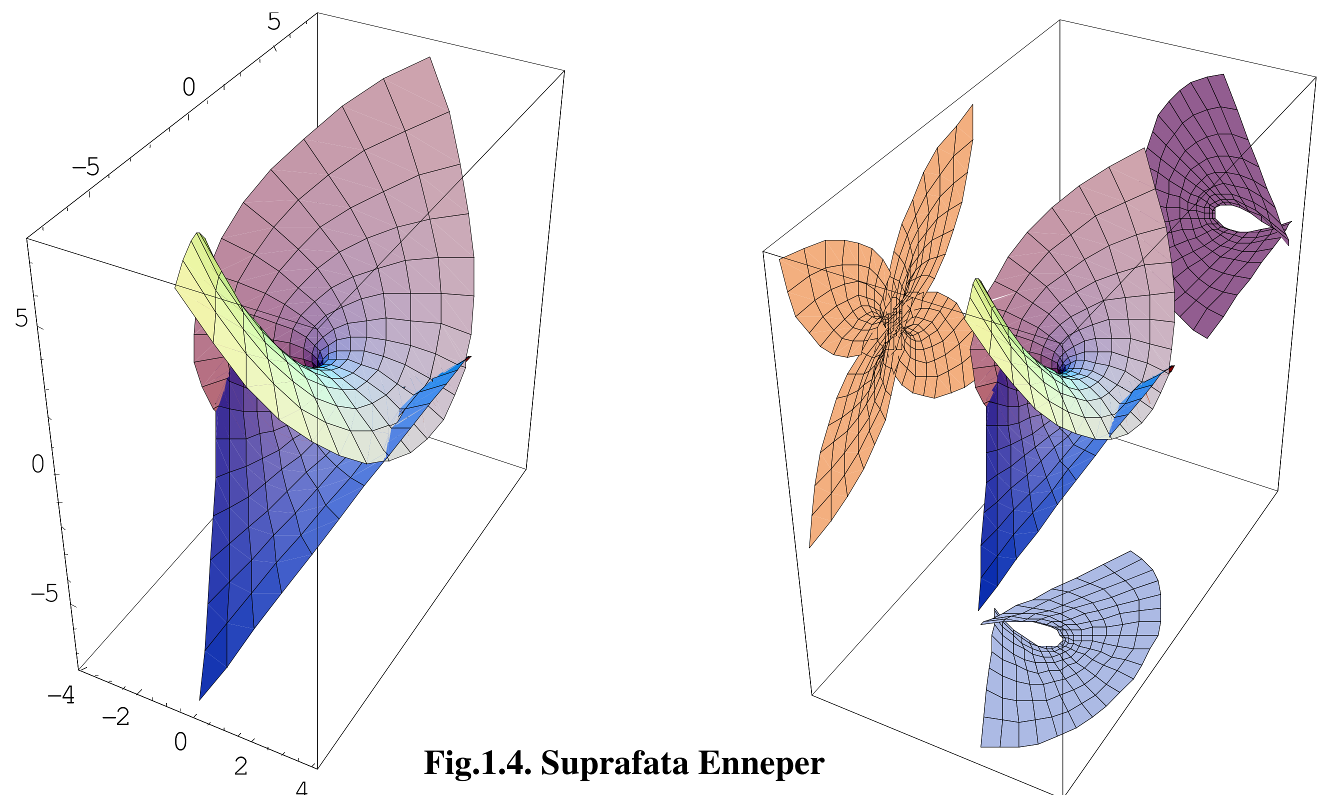


Fig.1.4. Suprafata Enneper

rotatie cu $\dfrac{\pi}{2}$ in jurul axei Oy

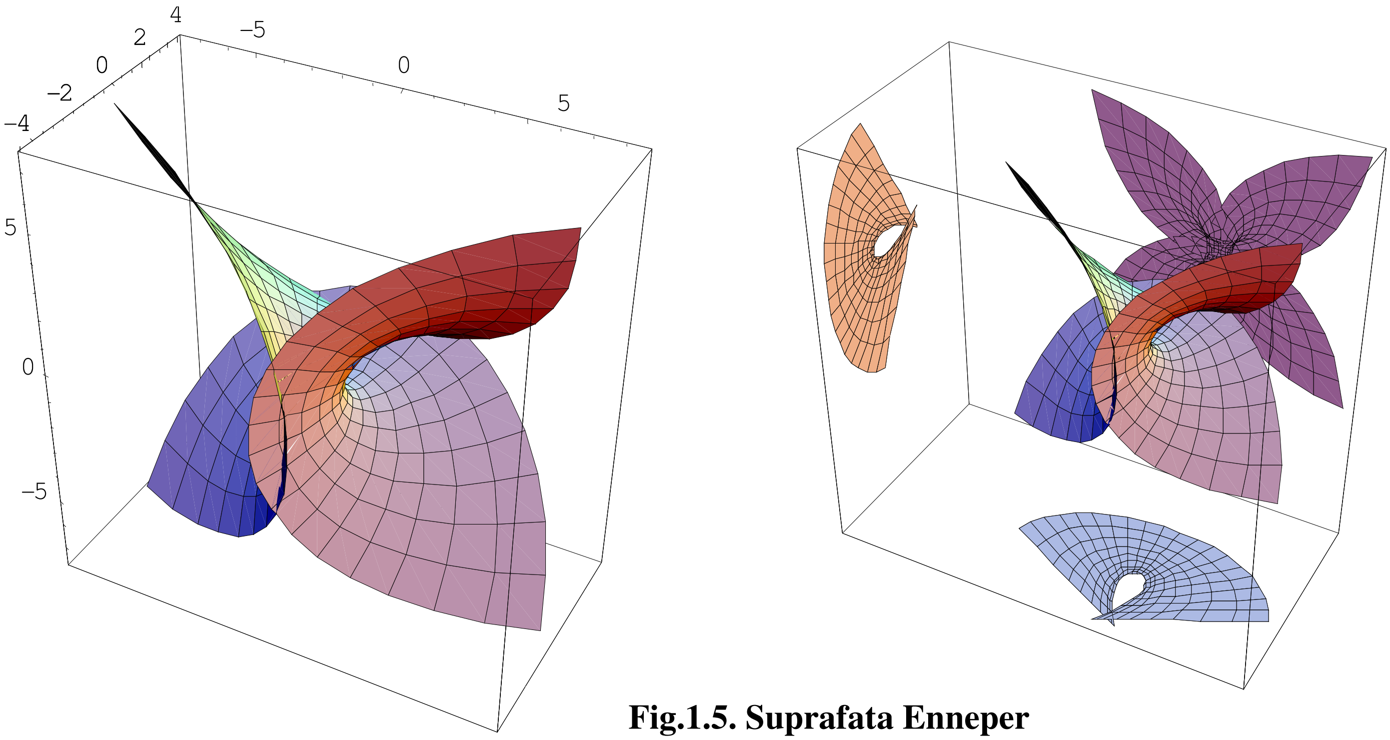


Fig.1.5. Suprafata Enneper
rotatie cu $-\dfrac{\pi}{2}$ in jurul axei Ox

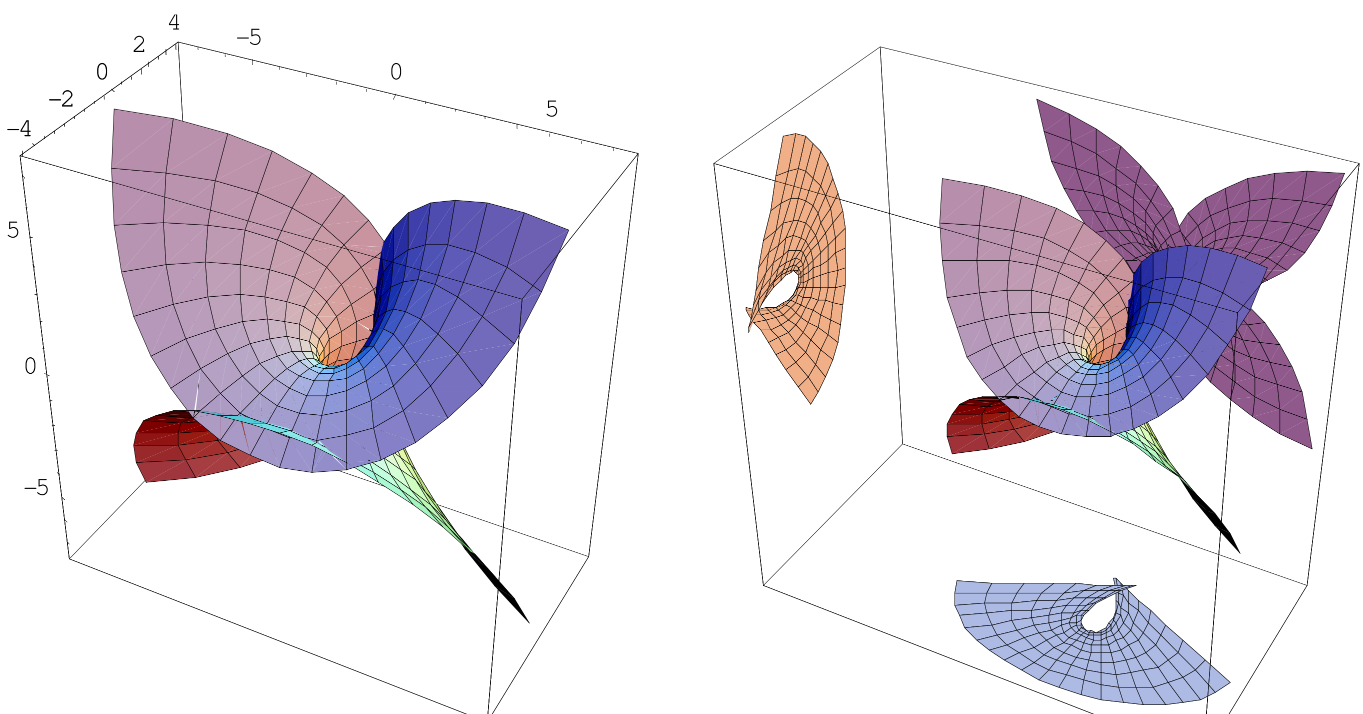


Fig.1.6. Suprafata Enneper
rotatie cu $\dfrac{\pi}{2}$ in jurul axei Ox

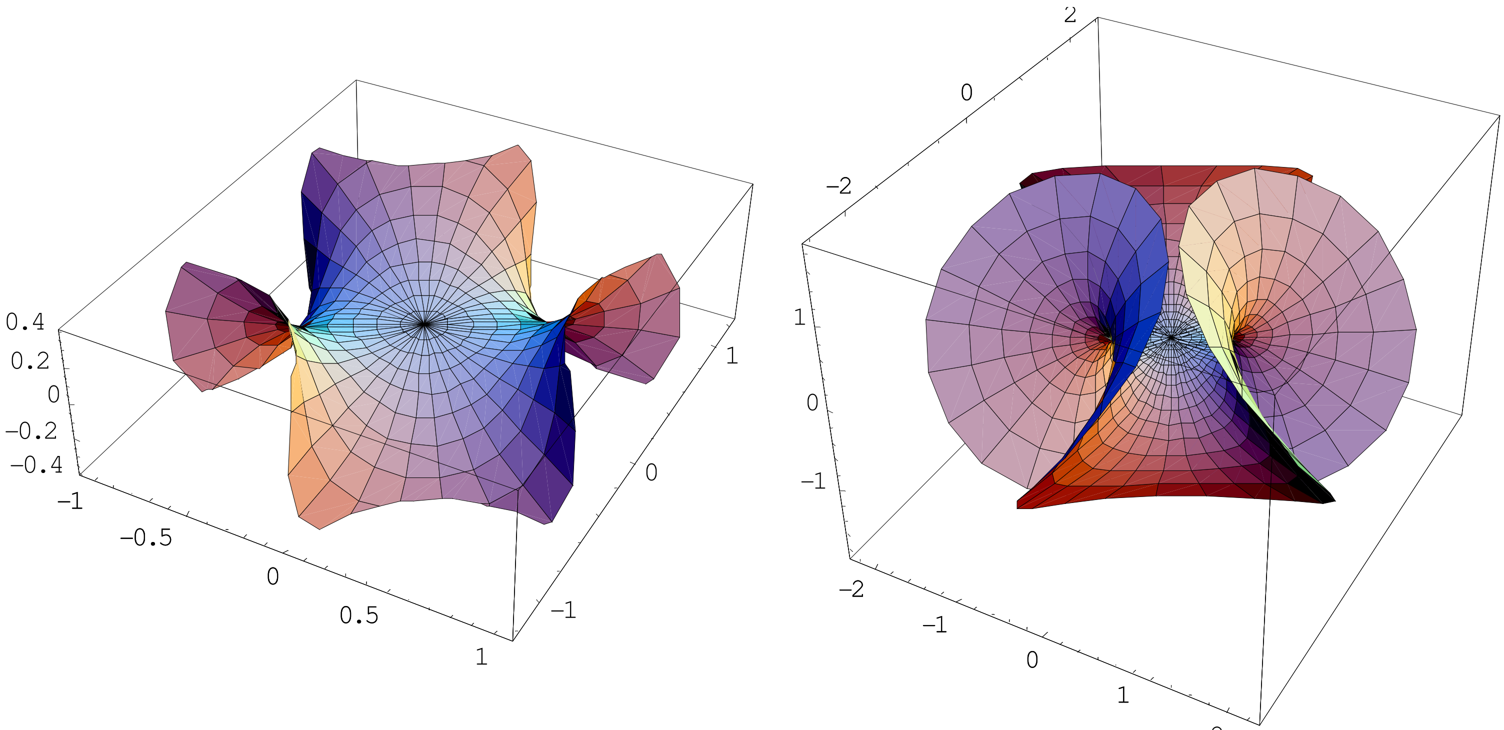


Fig.1.7. Suprafata Enneper de grad 2

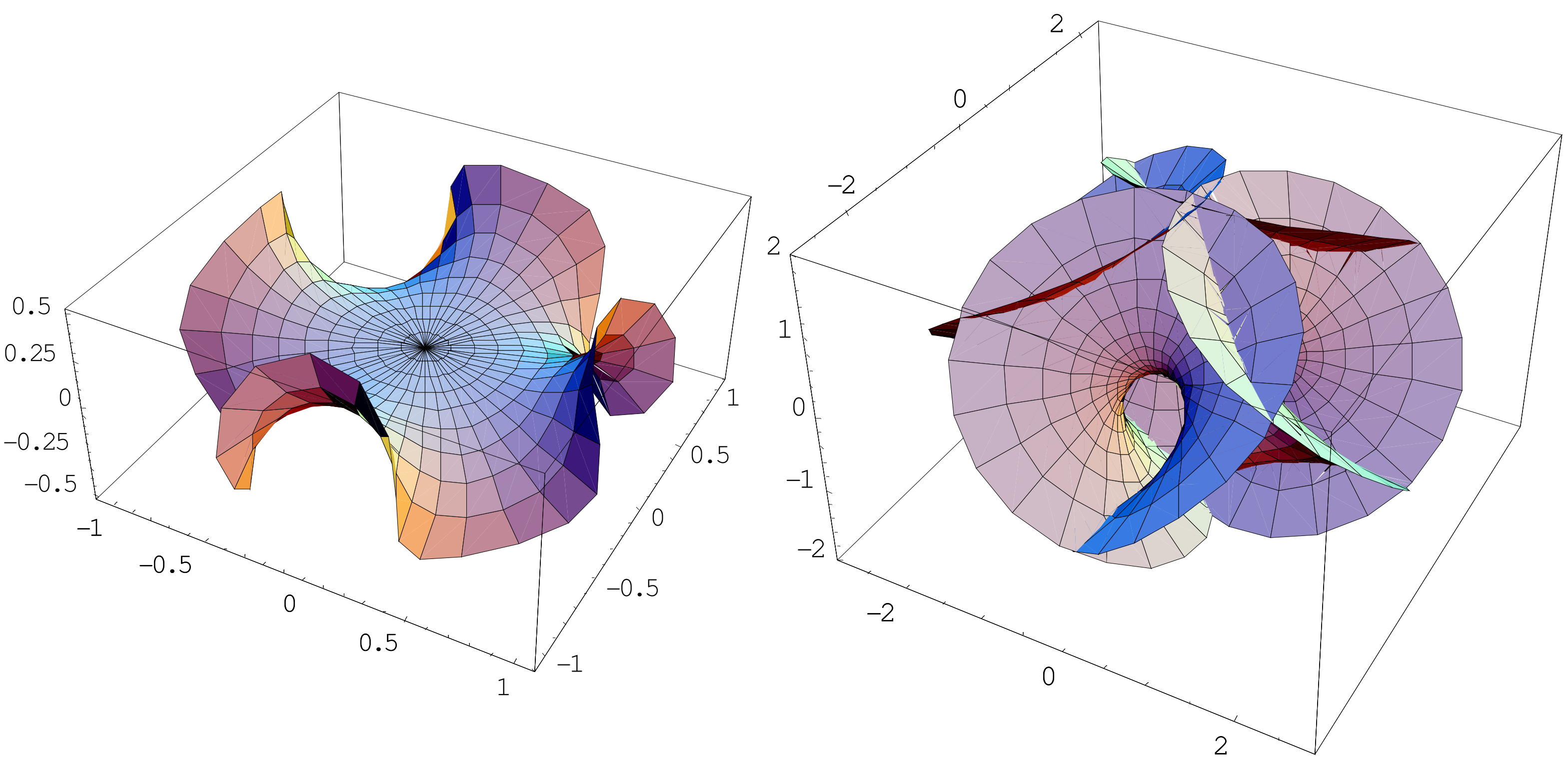


Fig.1.8. Suprafata Enneper de grad 3

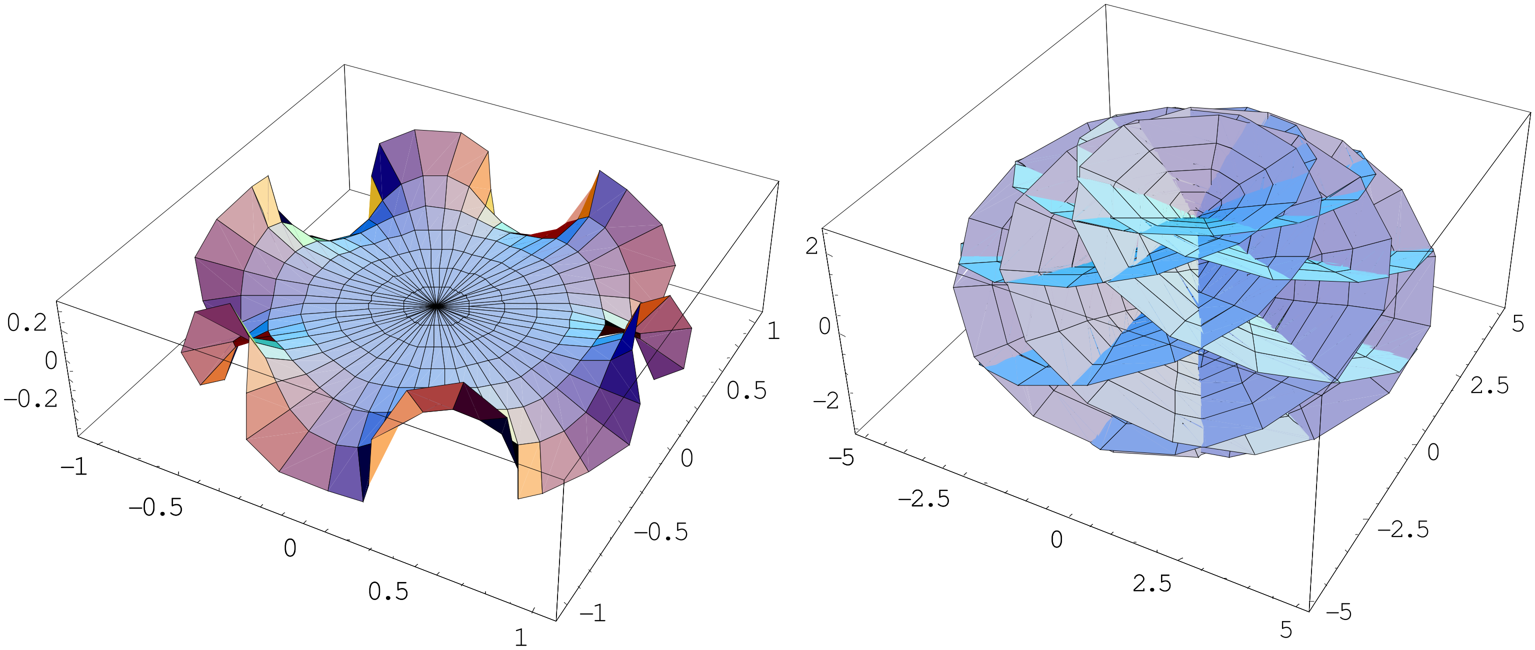


Fig.1.9. Suprafata Enneper de grad 5

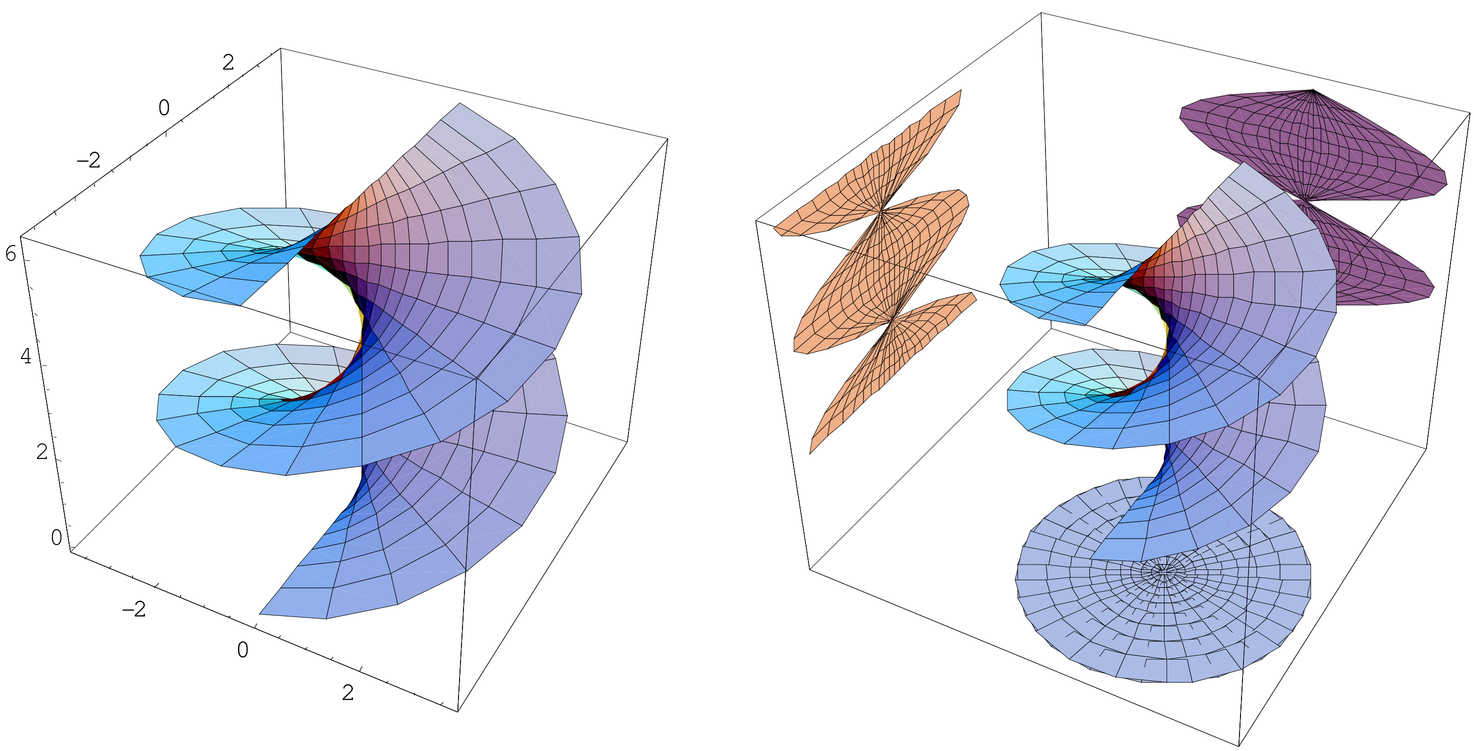


Fig.2.a. Elicoid (a=1,b=1)
elicoid-catenoid(0)

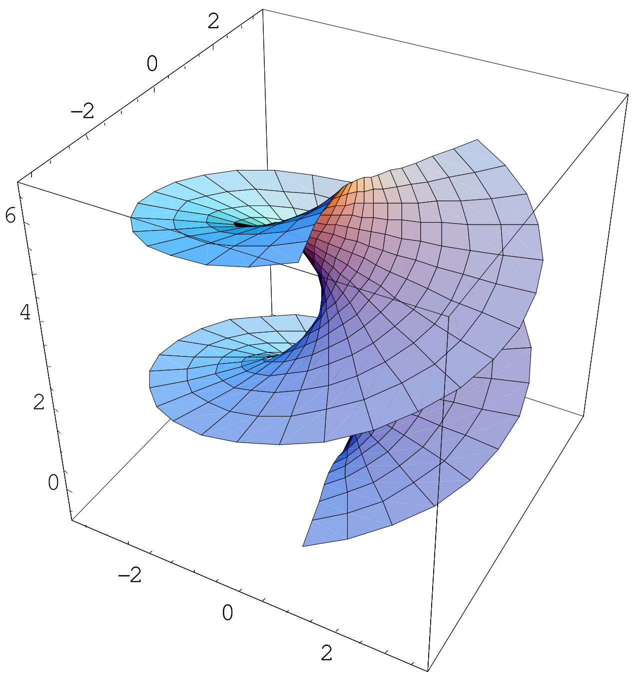


Fig.2.b.
elicoid-catenoid($\pi/10$)

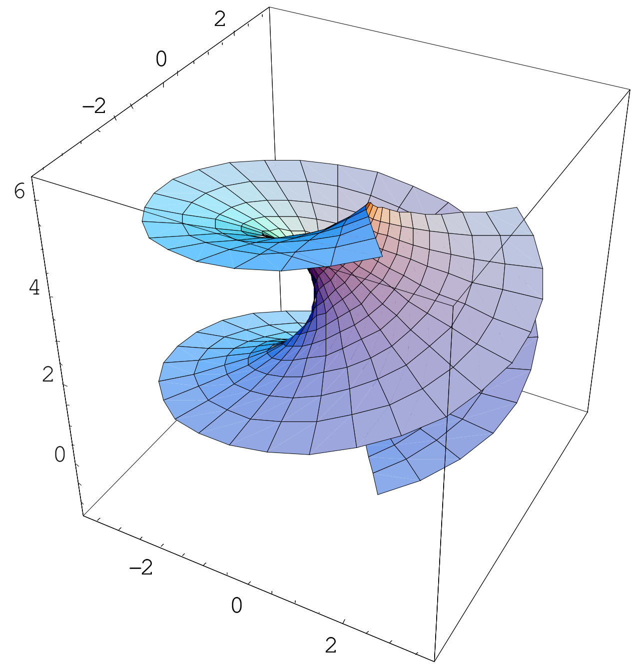


Fig.2.c.
elicoid-catenoid($\pi/5$)

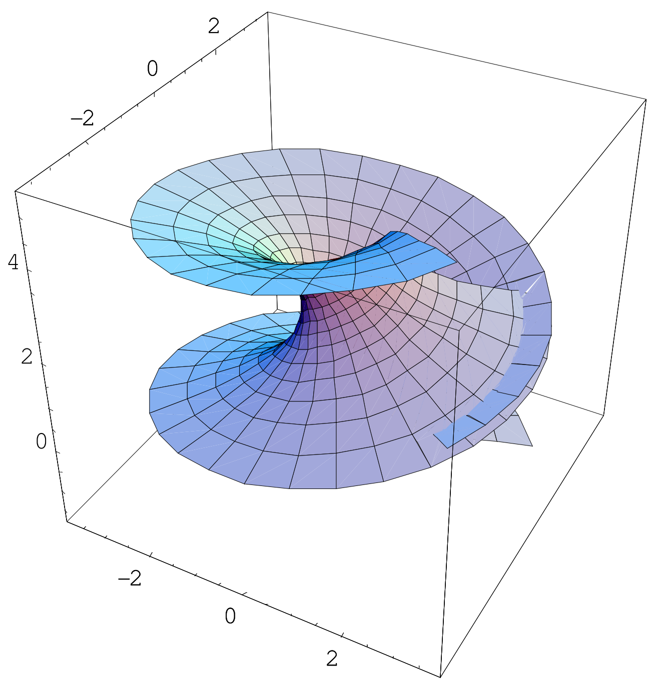


Fig. 2.d.
elicoid-catenoid($3\pi/10$)

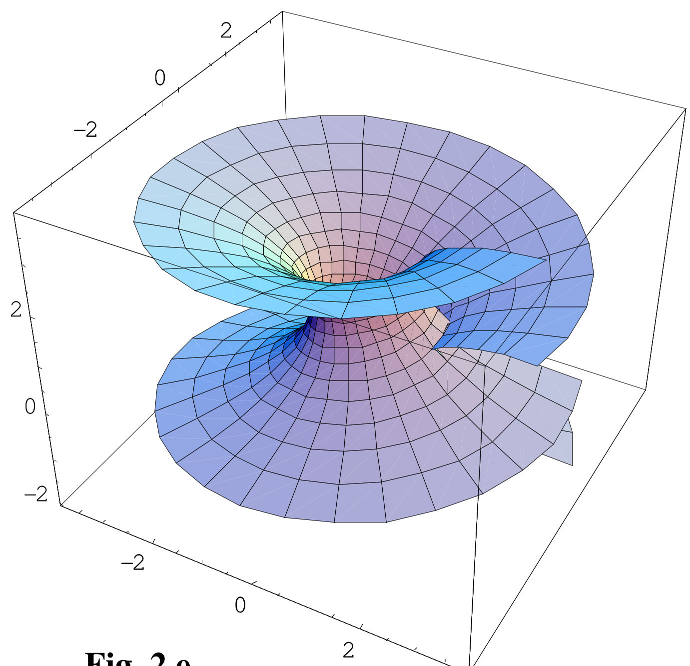


Fig. 2.e.
elicoid-catenoid($2\pi/5$)

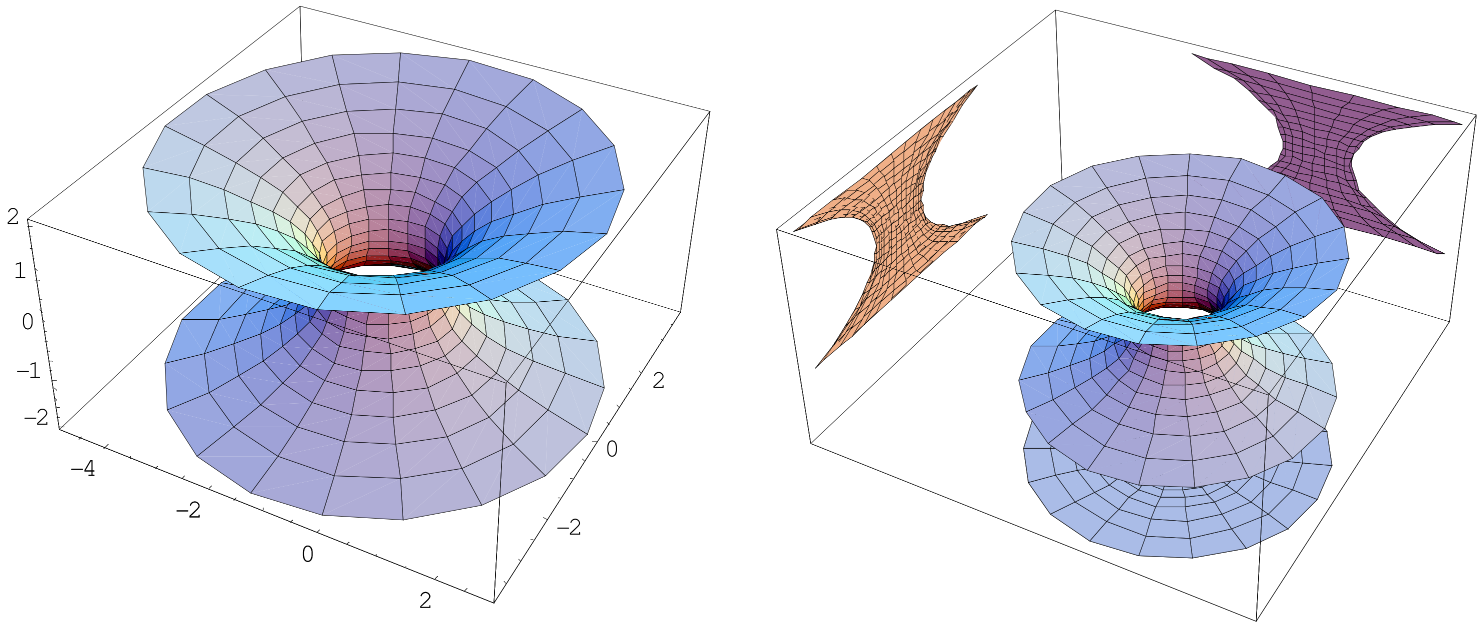


Fig.3. Catenoid (a=1)
elicoid-catenoid(0)

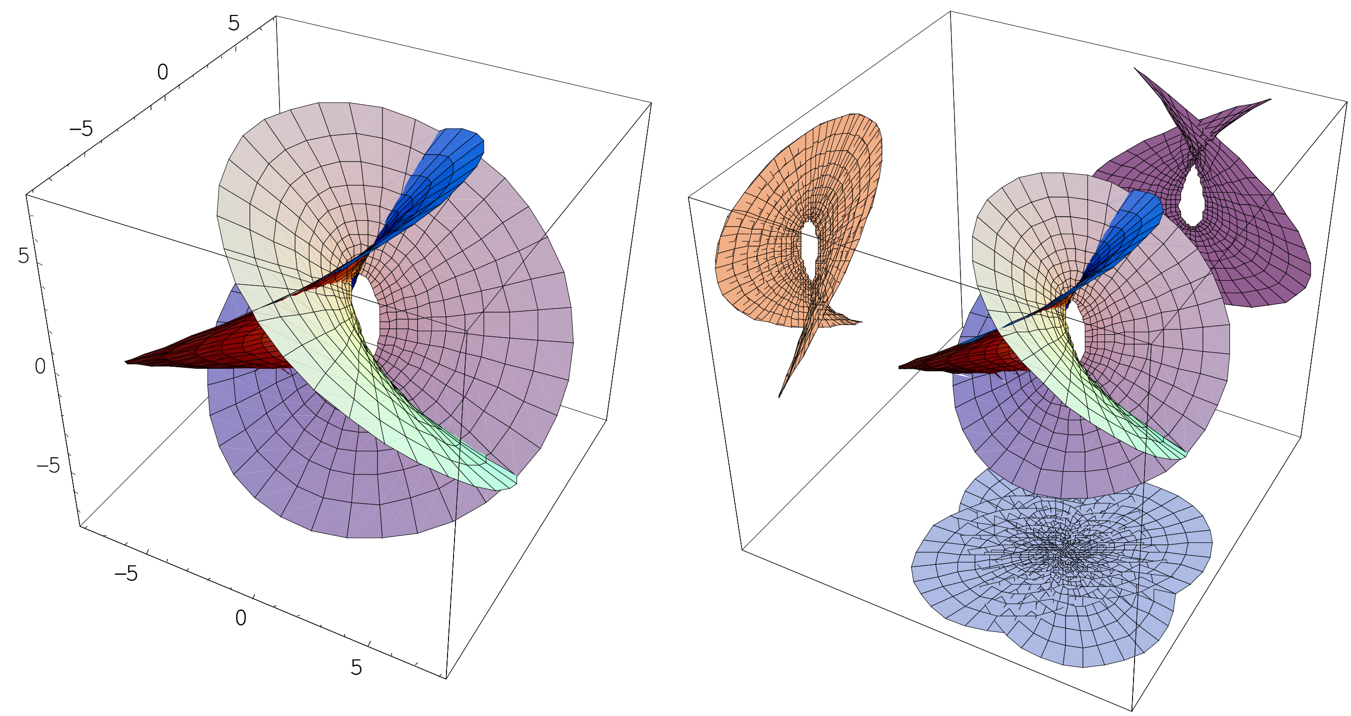
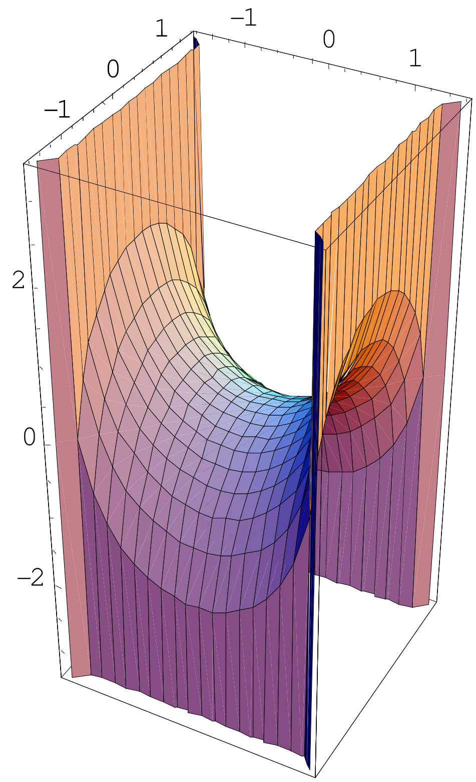


Fig. 4. Suprafata Henneberg

$$x(u,v) = \left(2sh u\cos v - \frac{2}{3}sh(3u)\cos(3v), 2sh u\sin v + \frac{2}{3}sh(3u)\sin(3v), 2ch(2u)\cos(2v)\right)$$



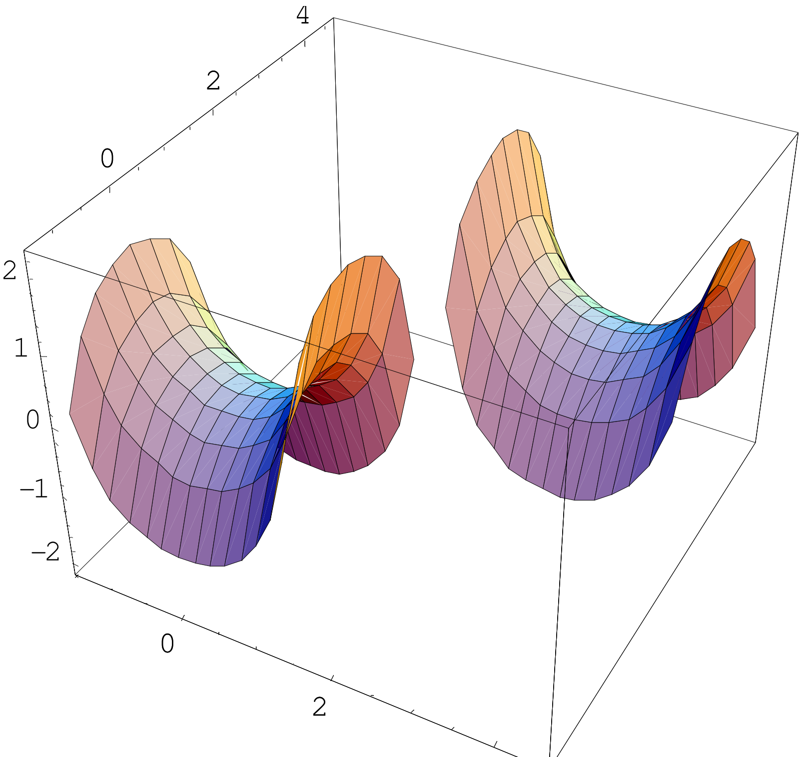
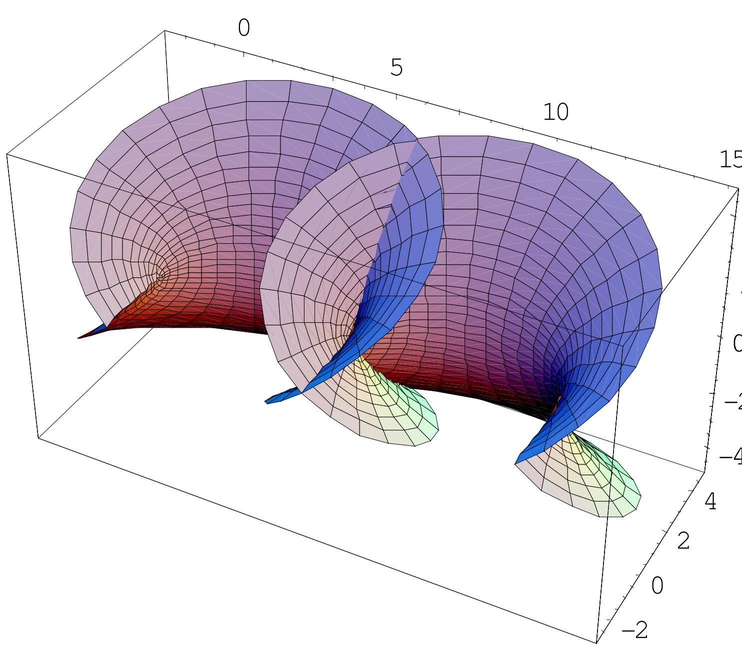


Fig.5. Suprafata Scherk

$$x(u,v) = \left(u, v, \frac{1}{a} \log \left(\frac{\cos(av)}{\cos(au)} \right) \right) \qquad (a = 1)$$



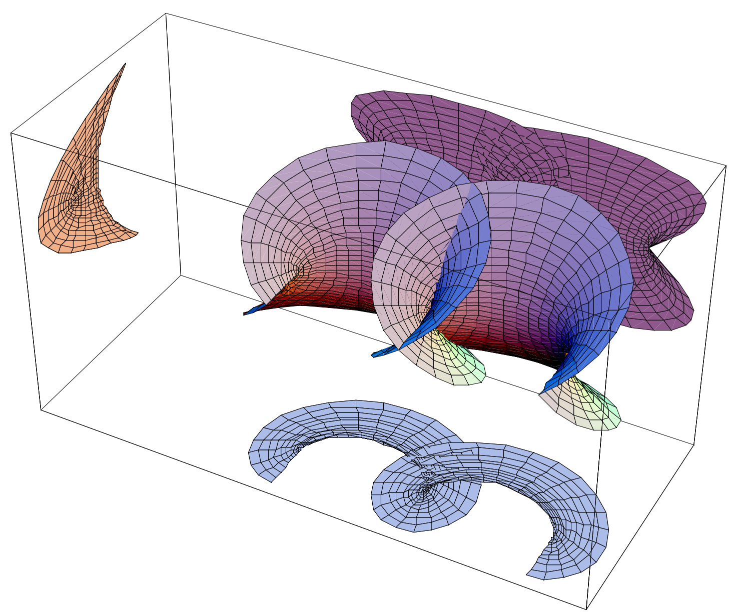


Fig. 6.a. Suprafata Catalan

$$x(u,v) = a\left(u - \sin u ch v, 1 - \cos u ch v, -4 \sin \frac{u}{2} sh \frac{v}{2} \right) \qquad (a = 1)$$

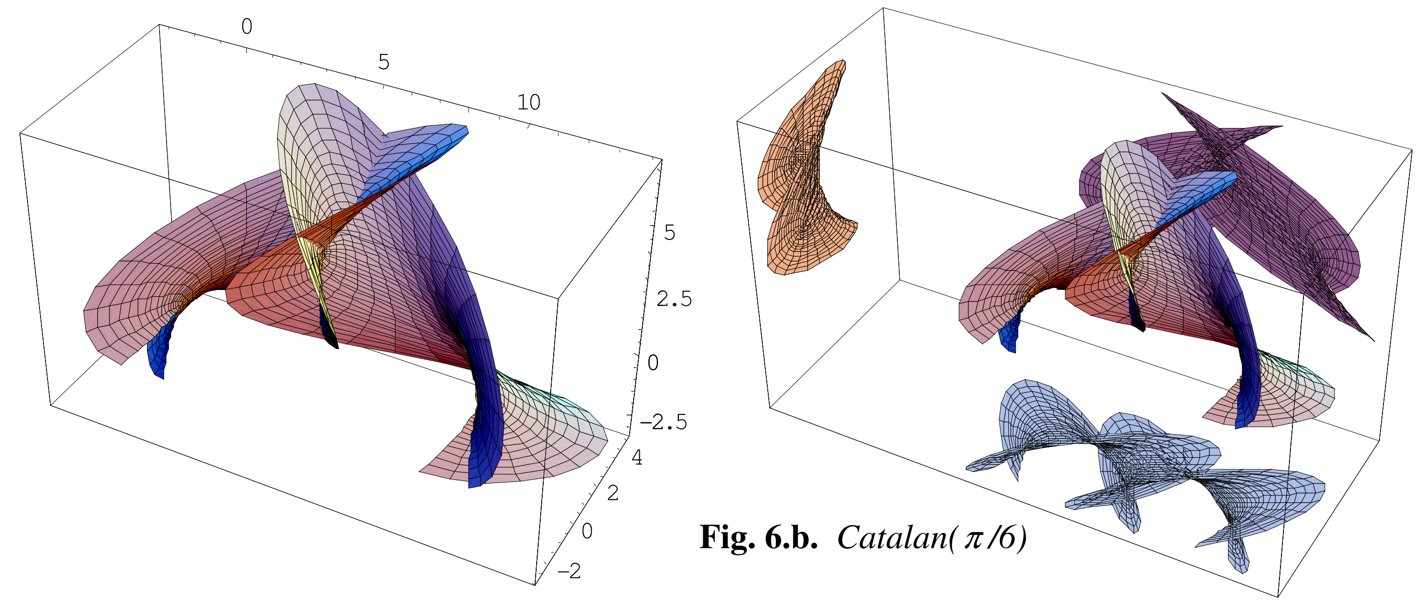


Fig. 6.b. *Catalan(π /6)*

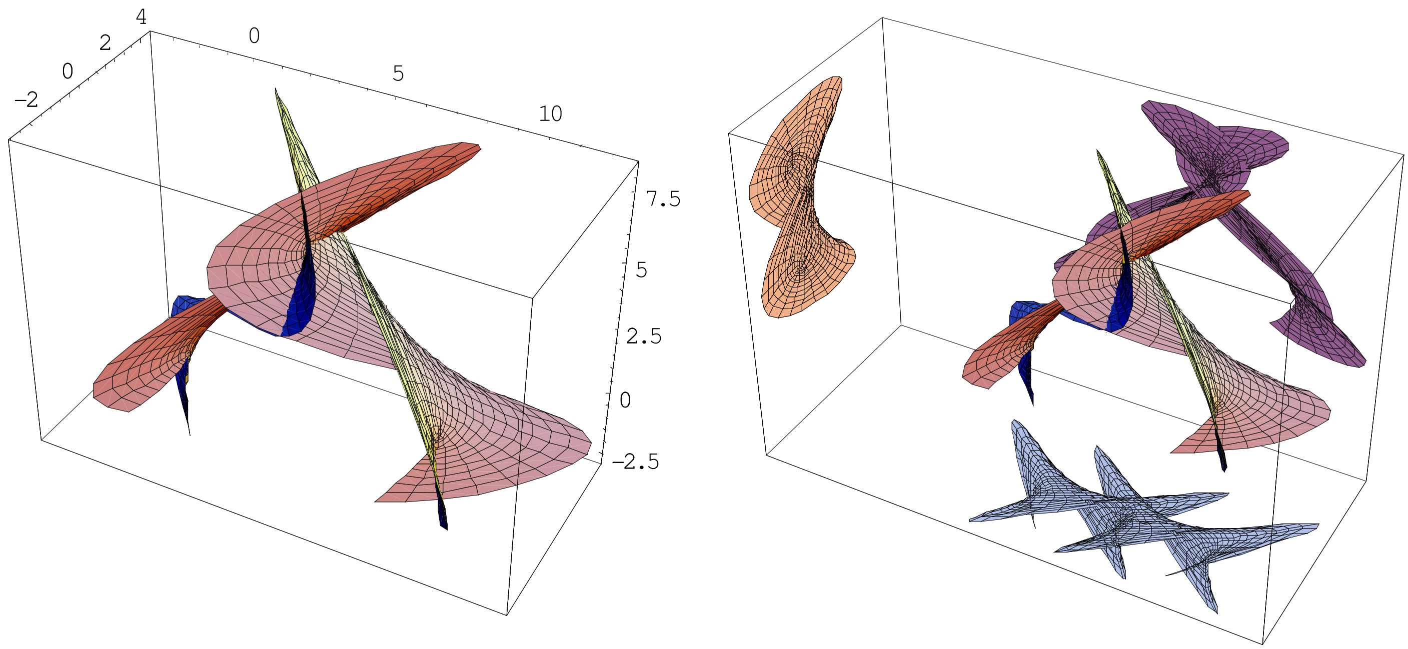


Fig. 6.c. *Catalan(π /4)*

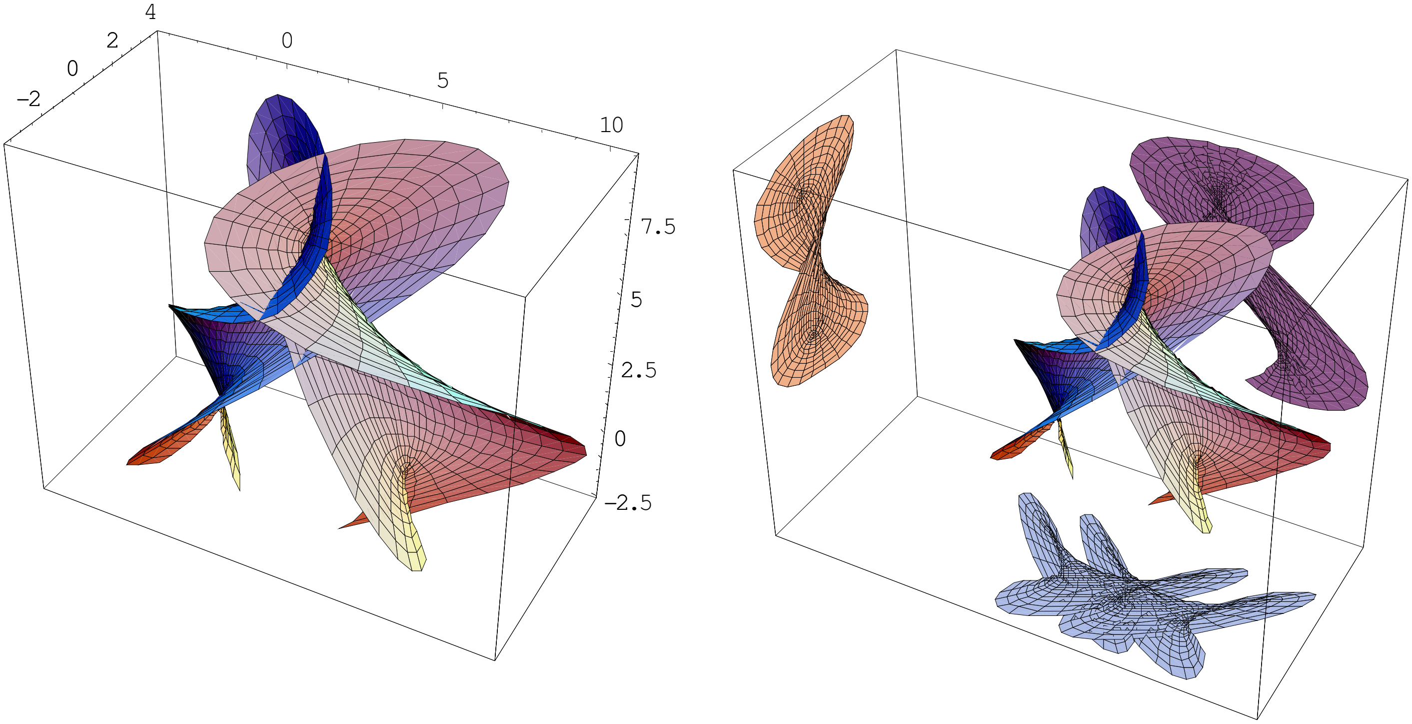


Fig. 6.d. *Catalan(π /3)*

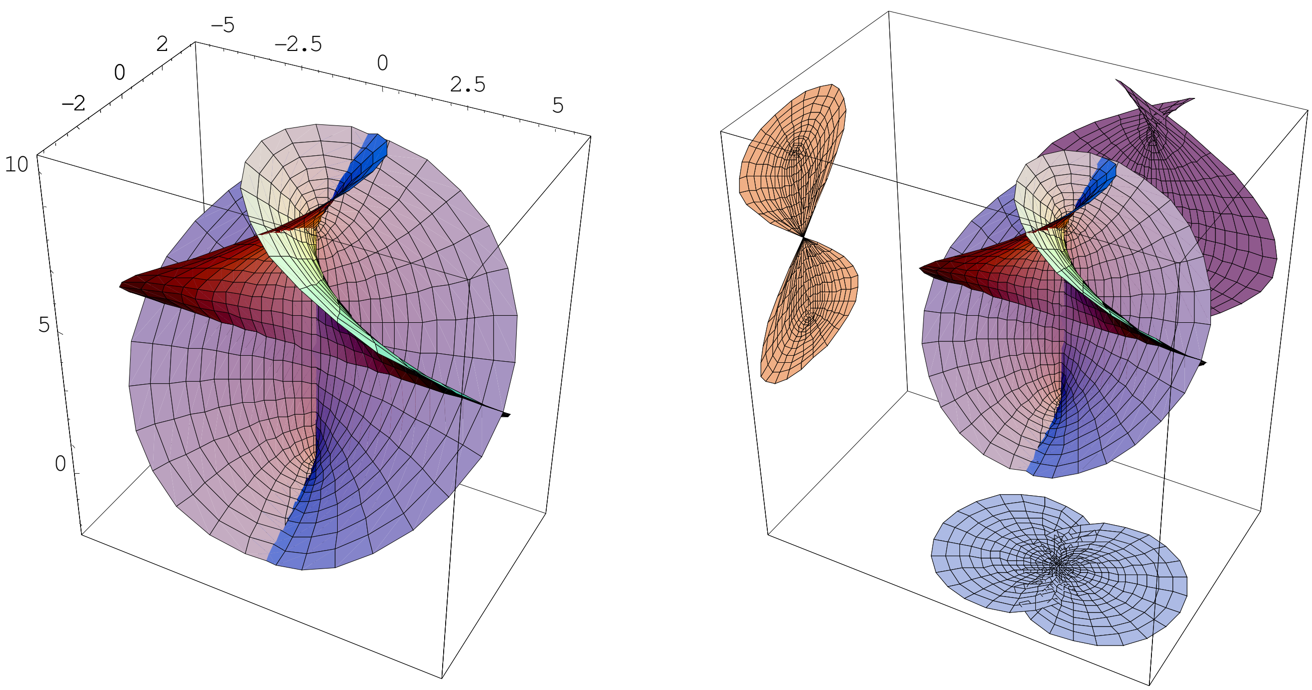


Fig.6.e. Conjugata suprafetei Catalan
Catalan(π /2)

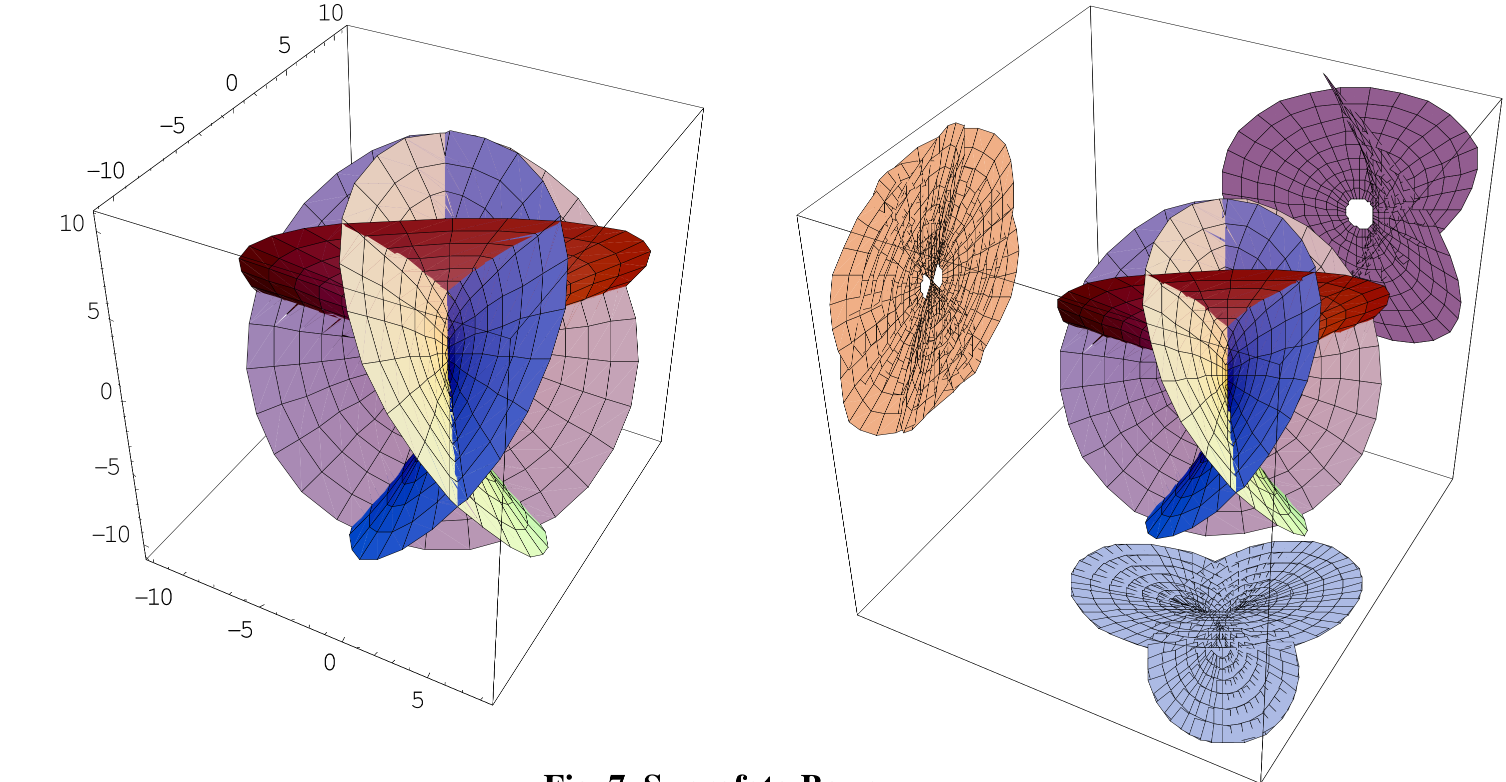


Fig. 7. Suprafata Bour

$$x(r,t) = \left(r\cos t - \frac{1}{2}r^2\cos(2t), -r\sin t - \frac{1}{2}r^2\sin(2t), \frac{4}{3}r^{3/2}\cos(\frac{3}{2}t) \right)$$

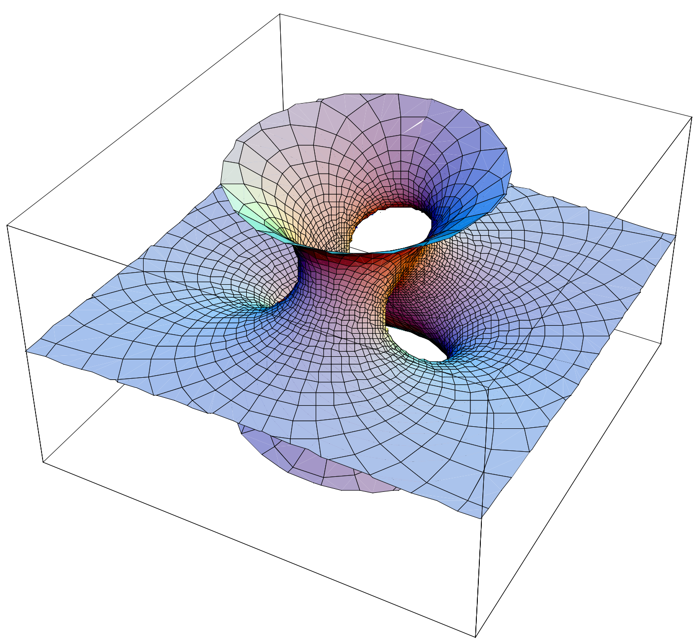


Fig.8. Suprafata Costa

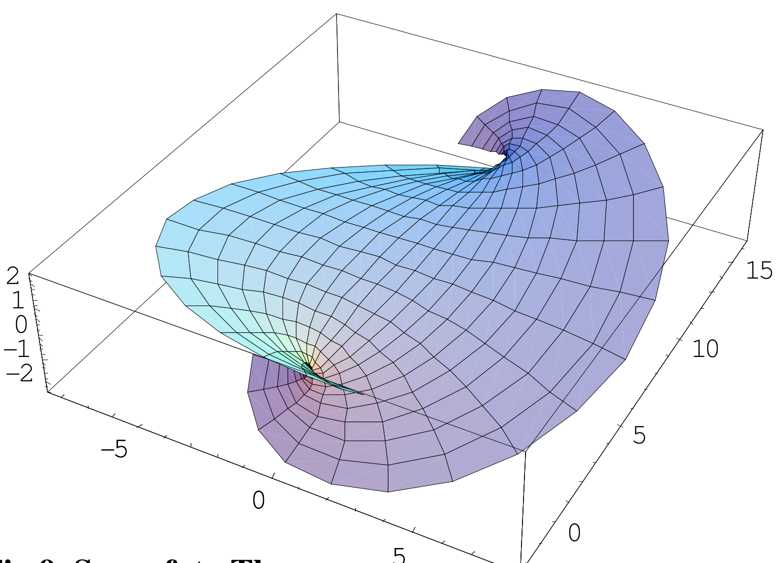
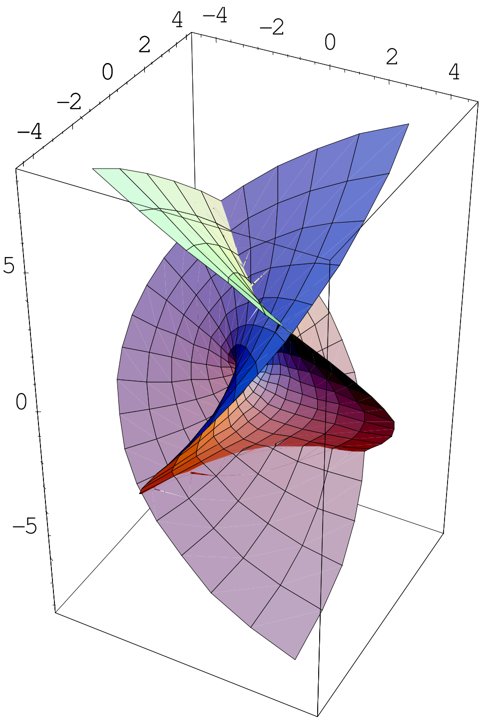


Fig.9. Suprafata Thomsen

$$x(u,v) = \left(\frac{b}{a}u + \frac{\sqrt{1+b^2}}{a^2}sh(au)\cos(av), \frac{\sqrt{1+b^2}}{a}v + \frac{b}{a^2}ch(au)\sin(av), \frac{1}{a^2}sh(au)ain(av) \right) \; (a = \mathrm{b} = 1)$$



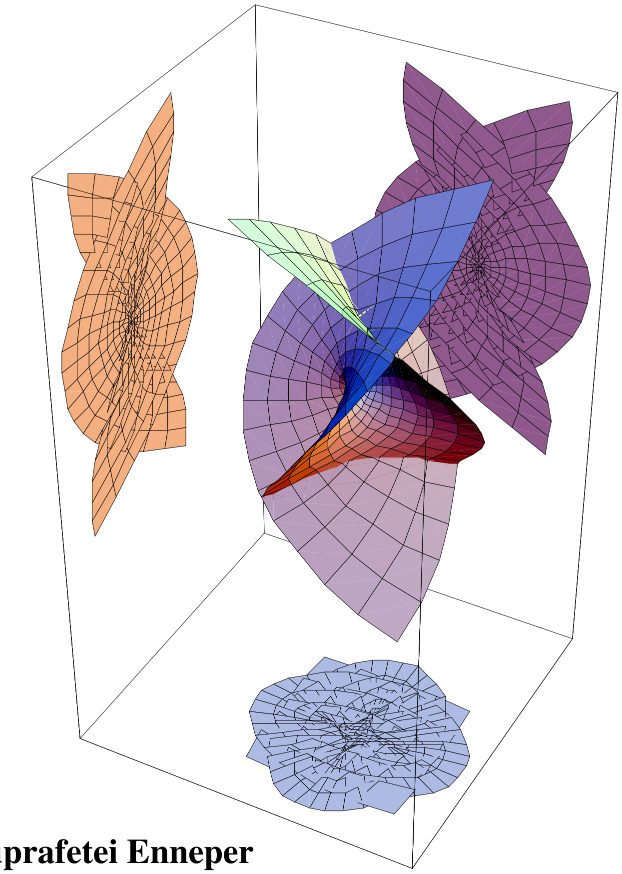
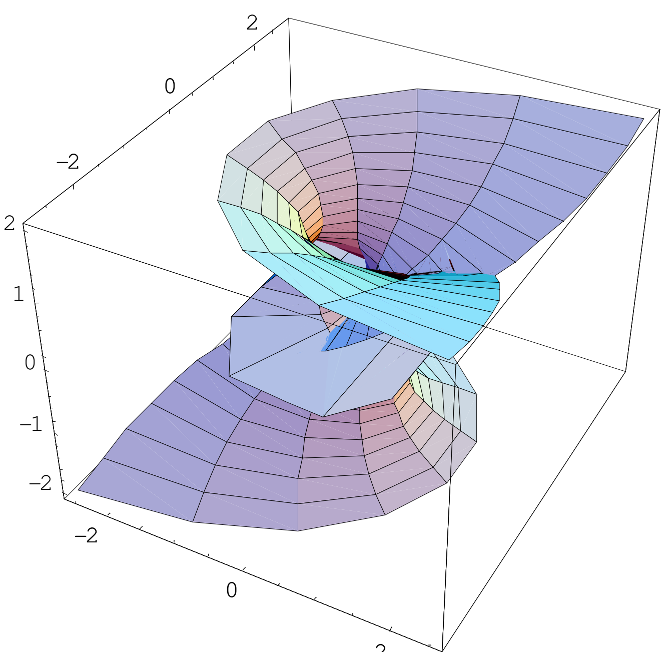


Fig. 10. Conjugata suprafetei Enneper



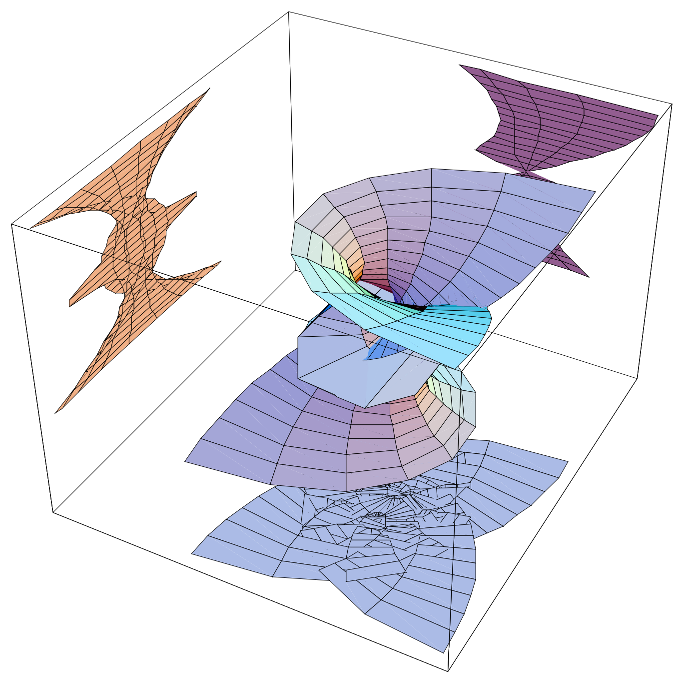


Fig.11.1. Suprafata Richmond

$$x(u,v)=\left(\frac{-3u-u^5+2u^3v^2+3uv^4}{6(u^2+v^2)},\frac{-3v+3u^4v-2u^2v^3+v^5}{6(u^2+v^2)},u\right)$$

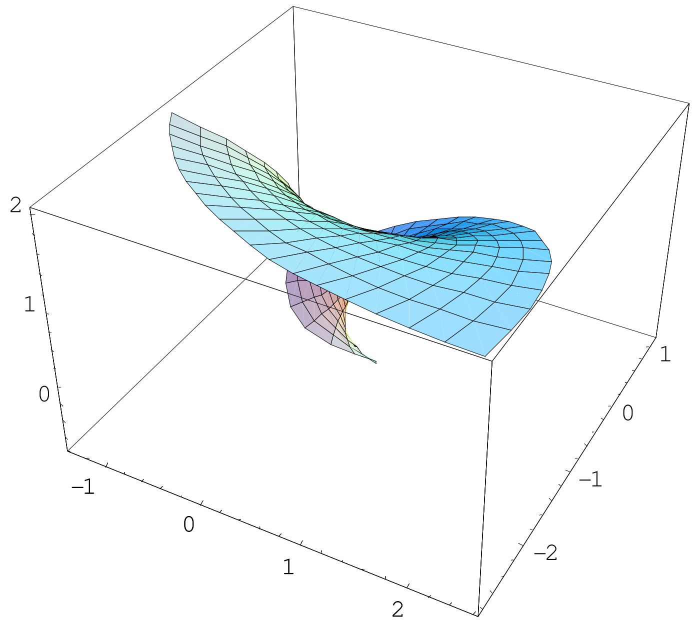


Fig.11.2. Suprafata Richmond

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