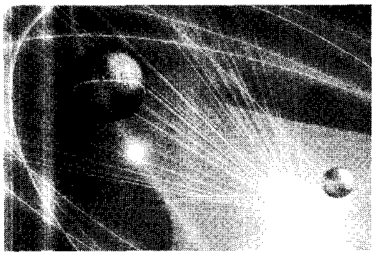

CHAPTER 11

Vector Functions and Curves

Introduction This chapter is concerned with functions of a single real variable that have *vector* values. Such functions can be thought of as parametric representations of curves, and we will examine them from both a *kinematic* point of view (involving position, velocity, and acceleration of a moving particle) and a *geometric* point of view (involving tangents, normals, curvature, and torsion). Finally, we will work through a simple derivation of Kepler's laws of planetary motion.

11.1 Vector Functions of One Variable

In this section we will examine several aspects of differential and integral calculus as applied to **vector-valued functions** of a single real variable. Such functions can be used to represent curves parametrically. It is natural to interpret a vector-valued function of the real variable t as giving the position, at time t, of a point or "particle" moving around in space. Derivatives of this *position vector* are then other vector-valued functions giving the velocity and acceleration of the particle. To motivate the study of vector functions we will consider such a vectorial description of motion in 3-space. Some of our examples will involve motion in the plane; in this case the third components of the vectors will be 0 and will be omitted.

If a particle moves around in 3-space, its motion can be described by giving the three coordinates of its position as functions of time t:

$$x = x(t), \qquad y = y(t), \qquad \text{and} \qquad z = z(t).$$

It is more convenient, however, to replace these three equations by a single vector equation,

$$\mathbf{r} = \mathbf{r}(t),$$

giving the position vector of the moving particle as a function of t. (Recall that the position vector of a point is the vector from the origin to that point.) In terms of the standard basis vectors **i**, **j**, and **k**, the position of the particle at time t is

$$\text{position:} \quad \mathbf{r} = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

As t increases, the particle moves along a *path*, a curve $\mathcal{C}$ in 3-space. If $z(t) = 0$, then $\mathcal{C}$ is a plane curve in the xy-plane. We assume that $\mathcal{C}$ is a *continuous curve*; the particle cannot instantaneously jump from one point to a distant point. This is equivalent to requiring that the component functions $x(t)$, $y(t)$, and $z(t)$ are continuous functions of t, and we therefore say that $\mathbf{r}(t)$ is a continuous vector function of t.

In the time interval from t to $t + \Delta t$ the particle moves from position $\mathbf{r}(t)$ to position $\mathbf{r}(t + \Delta t)$. Therefore, its **average velocity** is

$$\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t},$$

which is a vector parallel to the secant vector from $\mathbf{r}(t)$ to $\mathbf{r}(t+\Delta t)$. If the average velocity has a limit as $\Delta t \to 0$, then we say that $\mathbf{r}$ is **differentiable** at t, and we call the limit the (instantaneous) **velocity** of the particle at time t. We denote the velocity vector by $\mathbf{v}(t)$:

$$\text{velocity:}\quad \mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d}{dt}\mathbf{r}(t).$$

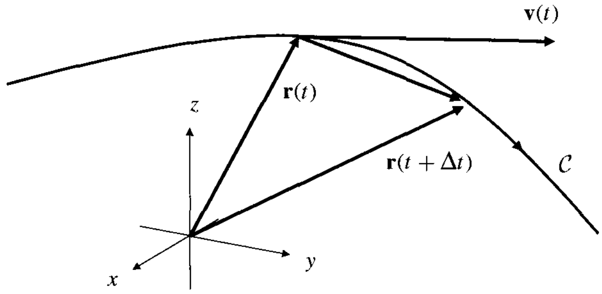

Figure 11.1 The velocity $\mathbf{v}(t)$ is the derivative of the position $\mathbf{r}(t)$ and is tangent to the path of motion at the point with position vector $\mathbf{r}(t)$

This velocity vector has direction tangent to the path $\mathcal{C}$ at the point $\mathbf{r}(t)$ (see Figure 11.1), and points in the direction of motion. The length of the velocity vector, $v(t) = |\mathbf{v}(t)|$, is called the **speed** of the particle:

$$\text{speed:}\quad v(t) = |\mathbf{v}(t)|.$$

Wherever the velocity vector exists, is continuous, and does not vanish, the path $\mathcal{C}$ is a **smooth** curve, that is, it has a continuously turning tangent line. The path may not be smooth at points where the velocity is zero, even if the components of the velocity vector are smooth functions of t.

Example 1 Consider the plane curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$. Its component functions t^3 and t^2 have continuous derivatives of all orders. However, the curve is not smooth at the origin ($t = 0$), where its velocity $\mathbf{v} = 3t^2\mathbf{i} + 2t\mathbf{j} = \mathbf{0}$. (See Figure 11.2.) The curve is smooth at all other points where $\mathbf{v}(t) \neq \mathbf{0}$.

The rules for addition and scalar multiplication of vectors imply that

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt}\\ &= \lim_{\Delta t \to 0}\left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\mathbf{i} + \frac{y(t+\Delta t) - y(t)}{\Delta t}\mathbf{j} + \frac{z(t+\Delta t) - z(t)}{\Delta t}\mathbf{k}\right)\\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.\end{aligned}$$

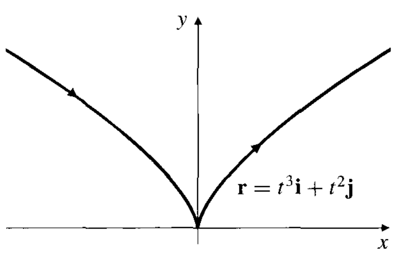

Figure 11.2 The components of $\mathbf{r}(t)$ are smooth functions of t, but the curve fails to be smooth at the origin, where $\mathbf{v} = \mathbf{0}$

Thus, the vector function $\mathbf{r}$ is differentiable at t if and only if its three scalar components, x, y, and z, are differentiable at t. In general, vector functions can be differentiated (or integrated) by differentiating (or integrating) their component functions, provided that the basis vectors with respect to which the components are taken are fixed in space and not changing with time.

Continuing our analysis of the moving particle, we define the **acceleration** of the particle to be the time derivative of the velocity:

$$\text{acceleration:} \quad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

Newton's Second Law of Motion asserts that this acceleration is proportional to, and *in the same direction as*, the force $\mathbf{F}$ causing the motion: if the particle has mass m, then the law is expressed by the *vector equation* $\mathbf{F} = m\mathbf{a}$.

Example 2 Describe the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Find the velocity and acceleration vectors for this curve at $(1, 1, 1)$.

Solution Since the scalar parametric equations for the curve are

$$x = t, \qquad y = t^2, \qquad \text{and} \qquad z = t^3,$$

which satisfy $y = x^2$ and $z = x^3$, the curve is the curve of intersection of the two cylinders $y = x^2$ and $z = x^3$. At any time t the velocity and acceleration vectors are given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k},$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} + 6t\mathbf{k}.$$

The point $(1, 1, 1)$ on the curve corresponds to $t = 1$, so the velocity and acceleration at that point are $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{a} = 2\mathbf{j} + 6\mathbf{k}$, respectively. ■

Example 3 Find the velocity, speed, and acceleration, and describe the motion of a particle whose position at time t is

$$\mathbf{r} = 3\cos\omega t\,\mathbf{i} + 4\cos\omega t\,\mathbf{j} + 5\sin\omega t\,\mathbf{k}.$$

Solution The velocity, speed, and acceleration are readily calculated:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -3\omega\sin\omega t\,\mathbf{i} - 4\omega\sin\omega t\,\mathbf{j} + 5\omega\cos\omega t\,\mathbf{k}$$
$$v = |\mathbf{v}| = 5\omega$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -3\omega^2\cos\omega t\,\mathbf{i} - 4\omega^2\cos\omega t\,\mathbf{j} - 5\omega^2\sin\omega t\,\mathbf{k} = -\omega^2\mathbf{r}.$$

Observe that $|\mathbf{r}| = 5$. Therefore, the path of the particle lies on the sphere with equation $x^2 + y^2 + z^2 = 25$. Since $x = 3\cos\omega t$ and $y = 4\cos\omega t$, the path also lies on the vertical plane $4x = 3y$. Hence, the particle moves around a circle of radius 5 centred at the origin and lying in the plane $4x = 3y$. Observe also that $\mathbf{r}$ is periodic with period $2\pi/\omega$. Therefore, the particle makes one revolution around the circle in time $2\pi/\omega$. The acceleration is always in the direction of $-\mathbf{r}$, that is, toward the origin. The term **centripetal acceleration** is used to describe such a "centre-seeking" acceleration. ■

Solution If the position of the particle at time t is $\mathbf{r}(t)$, then its acceleration is $d^2\mathbf{r}/dt^2$. The position of the particle can be found by solving the *initial-value problem*

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k}, \qquad \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{v}_0, \qquad \mathbf{r}(0) = \mathbf{r}_0.$$

We integrate the differential equation twice. Each integration introduces a *vector* constant of integration that we can determine from the given data by evaluating at $t = 0$:

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= -gt\mathbf{k} + \mathbf{v}_0 \\ \mathbf{r} &= -\frac{gt^2}{2}\mathbf{k} + \mathbf{v}_0 t + \mathbf{r}_0. \end{aligned}$$

The latter equation represents a parabola in the vertical plane passing through the point with position vector $\mathbf{r}_0$ and containing the vector $\mathbf{v}_0$. (See Figure 11.3.) The parabola has scalar parametric equations

$$\begin{aligned} x &= u_0 t + x_0, \\ y &= v_0 t + y_0, \\ z &= -\frac{gt^2}{2} + w_0 t + z_0, \end{aligned}$$

where $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ and $\mathbf{v}_0 = u_0\mathbf{i} + v_0\mathbf{j} + w_0\mathbf{k}$. ■

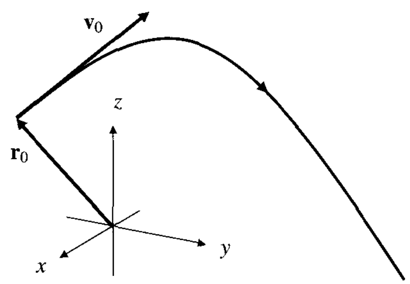

Figure 11.3 The path of a projectile fired from position $\mathbf{r}_0$ with velocity $\mathbf{v}_0$

Example 5 An object moves to the right along the plane curve $y = x^2$ with constant speed $v = 5$. Find the velocity and acceleration of the object when it is at the point $(1, 1)$.

Solution The position of the object at time t is

$$\mathbf{r} = x\mathbf{i} + x^2\mathbf{j},$$

where x, the x-coordinate of the object's position, is a function of t. The object's velocity, speed, and acceleration at time t are given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + 2x\frac{dx}{dt}\mathbf{j} = \frac{dx}{dt}(\mathbf{i} + 2x\mathbf{j}),$$
$$v = |\mathbf{v}| = \left|\frac{dx}{dt}\right|\sqrt{1 + (2x)^2} = \frac{dx}{dt}\sqrt{1 + 4x^2},$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}(\mathbf{i} + 2x\mathbf{j}) + 2\left(\frac{dx}{dt}\right)^2\mathbf{j}.$$

(In the speed calculation we used $|dx/dt| = dx/dt$ because the object is moving to the right.) We are given that the speed is constant; $v = 5$. Therefore,

$$\frac{dx}{dt} = \frac{5}{\sqrt{1 + 4x^2}}.$$

When $x = 1$, we have $dx/dt = 5/\sqrt{1 + 4} = \sqrt{5}$, so the velocity of the object at that point is $\mathbf{v} = \sqrt{5}\mathbf{i} + 2\sqrt{5}\mathbf{j}$. Now we can calculate

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{5}{\sqrt{1 + 4x^2}} = \left(\frac{d}{dx}\frac{5}{\sqrt{1 + 4x^2}}\right)\frac{dx}{dt}$$
$$= -\frac{5}{2(1 + 4x^2)^{3/2}}(8x)\frac{5}{\sqrt{1 + 4x^2}} = -\frac{100x}{(1 + 4x^2)^2}.$$

At $x = 1$, we have $d^2x/dt^2 = -4$. Thus, the acceleration at that point is $\mathbf{a} = -4(\mathbf{i} + 2\mathbf{j}) + 10\mathbf{j} = -4\mathbf{i} + 2\mathbf{j}$.

■

Remark Note that we used x as the parameter for the curve in the above example, so we could use t for time. If you want to analyze motion along a curve $\mathbf{r} = \mathbf{r}(t)$, where t is just a parameter, not necessarily time, then you will have to use a different symbol, say u, for time. The *physical velocity and acceleration* of a particle moving along the curve are then

$$\mathbf{v} = \frac{d\mathbf{r}}{du} = \frac{dt}{du}\frac{d\mathbf{r}}{dt} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{du} = \frac{d^2t}{du^2}\frac{d\mathbf{r}}{dt} + \left(\frac{dt}{du}\right)^2\frac{d^2\mathbf{r}}{dt^2}.$$

Be careful how you interpret t in a problem where time is meaningful.

Differentiating Combinations of Vectors

Vectors and scalars can be combined in a variety of ways to form other vectors or scalars. Vectors can be added and multiplied by scalars and can be factors in dot and cross products. Appropriate differentiation rules apply to all such combinations of vector and scalar functions; we summarize them in the following theorem.

THEOREM 1

Differentiation rules for vector functions

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable vector-valued functions, and let $\lambda(t)$ be a differentiable scalar-valued function. Then $\mathbf{u}(t) + \mathbf{v}(t)$, $\lambda(t)\mathbf{u}(t)$, $\mathbf{u}(t) \bullet \mathbf{v}(t)$, $\mathbf{u}(t) \times \mathbf{v}(t)$, and $\mathbf{u}\big(\lambda(t)\big)$ are differentiable, and

(a) $\frac{d}{dt}\left(\mathbf{u}(t)+\mathbf{v}(t)\right)=\mathbf{u}'(t)+\mathbf{v}'(t)$

(b) $\frac{d}{dt}\left(\lambda(t)\mathbf{u}(t)\right)=\lambda'(t)\mathbf{u}(t)+\lambda(t)\mathbf{u}'(t)$

(c) $\frac{d}{dt}\left(\mathbf{u}(t)\bullet\mathbf{v}(t)\right)=\mathbf{u}'(t)\bullet\mathbf{v}(t)+\mathbf{u}(t)\bullet\mathbf{v}'(t)$

(d) $\frac{d}{dt}\left(\mathbf{u}(t)\times\mathbf{v}(t)\right)=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

(e) $\frac{d}{dt}\left(\mathbf{u}(\lambda(t))\right)=\lambda'(t)\mathbf{u}'(\lambda(t)).$

Also, at any point where $\mathbf{u}(t)\neq 0$,

(f) $\frac{d}{dt}|\mathbf{u}(t)|=\frac{\mathbf{u}(t)\bullet\mathbf{u}'(t)}{|\mathbf{u}(t)|}.$

Remark Formulas (b), (c), and (d) are versions of the Product Rule. Formula (e) is a version of the Chain Rule. Formula (f) is also a case of the Chain Rule applied to $|\mathbf{u}|=\sqrt{\mathbf{u}\bullet\mathbf{u}}$. All have the obvious form. Note that the order of the factors is the same in the terms on both sides of the cross product formula (d). It is essential that the order be preserved because, unlike the dot product or the product of a vector with a scalar, the cross product is *not commutative.*

Remark The formula for the derivative of a cross product is a special case of that for the derivative of a 3×3 determinant. (See Section 10.3.) Since every term in the expansion of a determinant of any order is a product involving one element from each row (or column), the general Product Rule implies that the derivative of an $n\times n$ determinant whose elements are functions will be the sum of n such $n\times n$ determinants, each with the elements of one of the rows (or columns) differentiated. For the 3×3 case we have

$$\frac{d}{dt}\begin{vmatrix}a_{11}(t) & a_{12}(t) & a_{13}(t)\\ a_{21}(t) & a_{22}(t) & a_{23}(t)\\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix}=\begin{vmatrix}a'_{11}(t) & a'_{12}(t) & a'_{13}(t)\\ a_{21}(t) & a_{22}(t) & a_{23}(t)\\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix}$$
$$+\begin{vmatrix}a_{11}(t) & a_{12}(t) & a_{13}(t)\\ a'_{21}(t) & a'_{22}(t) & a'_{23}(t)\\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix}+\begin{vmatrix}a_{11}(t) & a_{12}(t) & a_{13}(t)\\ a_{21}(t) & a_{22}(t) & a_{23}(t)\\ a'_{31}(t) & a'_{32}(t) & a'_{33}(t)\end{vmatrix}.$$

Example 6 Show that the speed of a moving particle remains constant over an interval of time if and only if the acceleration is perpendicular to the velocity throughout that interval.

Solution Since $\left(v(t)\right)^2=\mathbf{v}(t)\bullet\mathbf{v}(t)$, we have

$$2v(t)\frac{dv}{dt} = \frac{d}{dt}\big(v(t)\big)^2 = \frac{d}{dt}\big(\mathbf{v}(t) \bullet \mathbf{v}(t)\big)$$
$$= \mathbf{a}(t) \bullet \mathbf{v}(t) + \mathbf{v}(t) \bullet \mathbf{a}(t) = 2\mathbf{v}(t) \bullet \mathbf{a}(t).$$

If we assume that $v(t) \neq 0$, it follows that $dv/dt = 0$ if and only if $\mathbf{v} \bullet \mathbf{a} = 0$. The speed is constant if and only if the velocity is perpendicular to the acceleration.

Example 7 If $\mathbf{u}$ is three times differentiable, calculate and simplify the triple product derivative

$$\frac{d}{dt}\left(\mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right)\right).$$

Solution Using various versions of the Product Rule, we calculate

$$\frac{d}{dt}\left(\mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right)\right)$$
$$= \frac{d\mathbf{u}}{dt} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \bullet \left(\frac{d^2\mathbf{u}}{dt^2} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right)$$
$$= 0 + 0 + \mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right) = \mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right).$$

The first term vanishes because $d\mathbf{u}/dt$ is perpendicular to its cross product with another vector; the second term vanishes because of the cross product of identical vectors.

Exercises 11.1

In Exercises 1–14, find the velocity, speed, and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

1. $\mathbf{r} = \mathbf{i} + t\mathbf{j}$

2. $\mathbf{r} = t^2\mathbf{i} + \mathbf{k}$

3. $\mathbf{r} = t^2\mathbf{j} + t\mathbf{k}$

4. $\mathbf{r} = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$

5. $\mathbf{r} = t^2\mathbf{i} - t^2\mathbf{j} + \mathbf{k}$

6. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$

7. $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$

8. $\mathbf{r} = a\cos\omega t\,\mathbf{i} + b\mathbf{j} + a\sin\omega t\,\mathbf{k}$

9. $\mathbf{r} = 3\cos t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 5\sin t\,\mathbf{k}$

10. $\mathbf{r} = 3\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + t\mathbf{k}$

11. $\mathbf{r} = ae^t\mathbf{i} + be^t\mathbf{j} + ce^t\mathbf{k}$

12. $\mathbf{r} = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$

13. $\mathbf{r} = e^{-t}\cos(e^t)\mathbf{i} + e^{-t}\sin(e^t)\mathbf{j} - e^t\mathbf{k}$

14. $\mathbf{r} = a\cos t\sin t\,\mathbf{i} + a\sin^2 t\,\mathbf{j} + a\cos t\,\mathbf{k}$

15. A particle moves around the circle $x^2 + y^2 = 25$ at constant speed, making one revolution in 2 s. Find its acceleration when it is at $(3, 4)$.

16. A particle moves to the right along the curve $y = 3/x$. If its speed is 10 when it passes through the point $\left(2, \frac{3}{2}\right)$, what is its velocity at that time?

17. A point P moves along the curve of intersection of the cylinder $z = x^2$ and the plane $x + y = 2$ in the direction of increasing y with constant speed $v = 3$. Find the velocity of P when it is at $(1, 1, 1)$.

18. An object moves along the curve $y = x^2$, $z = x^3$, with constant vertical speed $dz/dt = 3$. Find the velocity and acceleration of the object when it is at the point $(2, 4, 8)$.

19. A particle moves along the curve $\mathbf{r} = 3u\mathbf{i} + 3u^2\mathbf{j} + 2u^3\mathbf{k}$ in the direction corresponding to increasing u and with a constant speed of 6. Find the velocity and acceleration of the particle when it is at the point $(3, 3, 2)$.

20. A particle moves along the curve of intersection of the cylinders $y = -x^2$ and $z = x^2$ in the direction in which x increases. (All distances are in centimetres.) At the instant when the particle is at the point $(1, -1, 1)$, its speed is 9 cm/s, and that speed is increasing at a rate of 3 cm/s^2. Find the velocity and acceleration of the particle at that instant.

in Theorem 1(c).

23. Verify the formula for the derivative of a 3×3 determinant in the second remark following Theorem 1. Use this formula to verify the formula for the derivative of the cross product in Theorem 1.

24. If the position and velocity vectors of a moving particle are always perpendicular, show that the path of the particle lies on a sphere.

25. Generalize the previous problem to the case where the velocity of the particle is always perpendicular to the line joining the particle to a fixed point P_0.

26. What can be said about the motion of a particle at a time when its position and velocity satisfy $\mathbf{r} \bullet \mathbf{v} > 0$? What can be said when $\mathbf{r} \bullet \mathbf{v} < 0$?

In Exercises 27–32, assume that the vector functions encountered have continuous derivatives of all required orders.

27. Show that $\dfrac{d}{dt}\left(\dfrac{d\mathbf{u}}{dt} \times \dfrac{d^2\mathbf{u}}{dt^2}\right) = \dfrac{d\mathbf{u}}{dt} \times \dfrac{d^3\mathbf{u}}{dt^3}$.

28. Write the Product Rule for $\dfrac{d}{dt}\Big(\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\Big)$.

29. Write the Product Rule for $\dfrac{d}{dt}\Big(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})\Big)$.

32. Expand and simplify: $\dfrac{d}{dt}\Big((\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}'')\Big)$.

33. If at all times t the position and velocity vectors of a moving particle satisfy $\mathbf{v}(t) = 2\mathbf{r}(t)$, and if $\mathbf{r}(0) = \mathbf{r}_0$, find $\mathbf{r}(t)$ and the acceleration $\mathbf{a}(t)$. What is the path of motion?

◈ **34.** Verify that $\mathbf{r} = \mathbf{r}_0 \cos(\omega t) + (\mathbf{v}_0/\omega)\sin(\omega t)$ satisfies the initial-value problem

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}, \qquad \mathbf{r}'(0) = \mathbf{v}_0, \qquad \mathbf{r}(0) = \mathbf{r}_0.$$

(It is the unique solution.) Describe the path $\mathbf{r}(t)$. What is the path if $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$?

◈ **35. (Free fall with air resistance)** A projectile falling under gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - c\frac{d\mathbf{r}}{dt},$$

where c is a positive constant. If $\mathbf{r} = \mathbf{r}_0$ and $d\mathbf{r}/dt = \mathbf{v}_0$ at time $t = 0$, find $\mathbf{r}(t)$. (*Hint:* let $\mathbf{w} = e^{ct}(d\mathbf{r}/dt)$.) Show that the solution approaches that of the projectile problem given in this section as $c \to 0$.

11.2 Some Applications of Vector Differentiation

Many interesting problems in mechanics involve the differentiation of vector functions. This section is devoted to a brief discussion of a few of these.

Motion Involving Varying Mass

The **momentum p** of a moving object is the product of its (scalar) mass m and its (vector) velocity $\mathbf{v}$; $\mathbf{p} = m\mathbf{v}$. Newton's Second Law of Motion states that the rate of change of *momentum* is equal to the external force acting on the object:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\Big(m\mathbf{v}\Big).$$

It is only when the mass of the object remains constant that this law reduces to the more familiar $\mathbf{F} = m\mathbf{a}$. When mass is changing you must deal with momentum rather than acceleration.

Example 1 **(The changing velocity of a rocket)** A rocket accelerates by burning its onboard fuel. If the exhaust gases are ejected with constant velocity $\mathbf{v}_e$ *relative to the rocket*, and if the rocket ejects $p\%$ of its initial mass while its engines are firing, by what amount will the velocity of the rocket change? Assume the rocket is in deep space so that gravitational and other external forces acting on it can be neglected.

Solution Since the rocket is not acted on by any external forces (i.e., $\mathbf{F} = \mathbf{0}$), Newton's law implies that the total momentum of the rocket and its exhaust gases will remain constant. At time t the rocket has mass $m(t)$ and velocity $\mathbf{v}(t)$. At time $t + \Delta t$ the rocket's mass is $m + \Delta m$ (where $\Delta m < 0$), its velocity is $\mathbf{v} + \Delta\mathbf{v}$, and the mass $-\Delta m$ of exhaust gases has escaped with velocity $\mathbf{v} + \mathbf{v}_e$ (relative to a coordinate system fixed in space). Equating total momenta at t and $t + \Delta t$ we obtain

$$(m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) + (-\Delta m)(\mathbf{v} + \mathbf{v}_e) = m\mathbf{v}.$$

Simplifying this equation and dividing by Δt gives

$$(m + \Delta m)\frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta m}{\Delta t}\mathbf{v}_e,$$

and, on taking the limit as $\Delta t \to 0$,

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e.$$

Suppose that the engine fires from $t = 0$ to $t = T$. By the Fundamental Theorem of Calculus, the velocity of the rocket will change by

$$\begin{aligned}\mathbf{v}(T) - \mathbf{v}(0) &= \int_0^T \frac{d\mathbf{v}}{dt}\,dt = \left(\int_0^T \frac{1}{m}\frac{dm}{dt}\,dt\right)\mathbf{v}_e \\ &= \Big(\ln m(T) - \ln m(0)\Big)\mathbf{v}_e = -\ln\left(\frac{m(0)}{m(T)}\right)\mathbf{v}_e.\end{aligned}$$

Since $m(0) > m(T)$, we have $\ln\big(m(0)/m(T)\big) > 0$ and, as was to be expected, the change in velocity of the rocket is in the opposite direction to the exhaust velocity $\mathbf{v}_e$. If $p\%$ of the mass of the rocket is ejected during the burn, then the velocity of the rocket will change by the amount $-\mathbf{v}_e \ln(100/(100 - p))$. ■

Remark It is interesting that this model places no restriction on how great a velocity the rocket can achieve, provided that a sufficiently large percentage of its initial mass is fuel. See Exercise 1 at the end of the section.

Circular Motion

The angular speed Ω of a rotating body is its rate of rotation measured in radians per unit time. For instance, a lighthouse lamp rotating at a rate of three revolutions per minute has an angular speed of $\Omega = 6\pi$ radians per minute. It is useful to represent the rate of rotation of a rigid body about an axis in terms of an **angular velocity** vector rather than just the scalar angular speed. The angular velocity vector, $\boldsymbol{\Omega}$, has

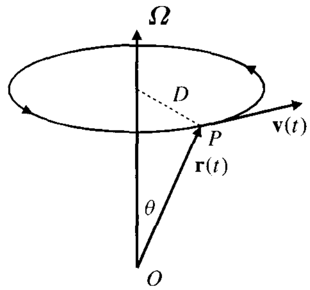

Figure 11.4 Rotation with angular velocity $\boldsymbol{\Omega}$: $\mathbf{v} = \boldsymbol{\Omega}\times\mathbf{r}$

magnitude equal to the angular speed, Ω, and direction along the axis of rotation such that if the extended right thumb points in the direction of $\boldsymbol{\Omega}$, then the fingers surround the axis in the direction of rotation.

If the origin of the coordinate system is on the axis of rotation, and $\mathbf{r} = \mathbf{r}(t)$ is the position vector at time t of a point P in the rotating body, then P moves around a circle of radius $D = |\mathbf{r}(t)|\sin\theta$, where θ is the (constant) angle between $\boldsymbol{\Omega}$ and $\mathbf{r}(t)$. (See Figure 11.4.) Thus, P travels a distance $2\pi D$ in time $2\pi/\Omega$, and its linear speed is

$$\frac{\text{distance}}{\text{time}} = \frac{2\pi D}{2\pi/\Omega} = \Omega D = |\boldsymbol{\Omega}||\mathbf{r}(t)|\sin\theta = |\boldsymbol{\Omega}\times\mathbf{r}(t)|.$$

Since the direction of $\boldsymbol{\Omega}$ was defined so that $\boldsymbol{\Omega}\times\mathbf{r}(t)$ would point in the direction of motion of P, the linear velocity of P at time t is given by

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \boldsymbol{\Omega}\times\mathbf{r}(t).$$

Example 2 The position vector $\mathbf{r}(t)$ of a moving particle P satisfies the initial-value problem

$$\begin{cases} \dfrac{d\mathbf{r}}{dt} = 2\mathbf{i}\times\mathbf{r}(t) \\ \mathbf{r}(0) = \mathbf{i} + 3\mathbf{j}. \end{cases}$$

Find $\mathbf{r}(t)$ and describe the motion of P.

Solution There are two ways to solve this problem. We will do it both ways.

Method I. By the discussion above, the given differential equation is consistent with rotation about the x-axis with angular velocity $2\mathbf{i}$, so that the angular speed is 2 and the motion is counterclockwise as seen from far out on the positive x-axis. Therefore, the particle P moves on a circle in a plane x = constant and centred on the x-axis. Since P is at $(1, 3, 0)$ at time $t = 0$, the plane of motion is $x = 1$ and the radius of the circle is 3. Therefore, the circle has a parametric equation of the form

$$\mathbf{r} = \mathbf{i} + 3\cos(\lambda t)\mathbf{j} + 3\sin(\lambda t)\mathbf{k}.$$

P travels once around this circle (2π radians) in time $t = 2\pi/\lambda$, so the angular speed is λ. Therefore, $\lambda = 2$ and the motion of the particle is given by

$$\mathbf{r} = \mathbf{i} + 3\cos(2t)\mathbf{j} + 3\sin(2t)\mathbf{k}.$$

Method II. Break the given vector differential equation into components:

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = 2\mathbf{i}\times(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -2z\mathbf{j} + 2y\mathbf{k}$$

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = -2z, \qquad \frac{dz}{dt} = 2y.$$

The first equation implies that x = constant. Since $x(0) = 1$, we have $x(t) = 1$ for all t. Differentiate the second equation with respect to t and substitute the third equation. This leads to the equation of simple harmonic motion for y,

$$\frac{d^2y}{dt^2} = -2\frac{dz}{dt} = -4y,$$

for which a general solution is

$$y = A\cos(2t) + B\sin(2t).$$

Thus, $z = -\frac{1}{2}(dy/dt) = A\sin(2t) - B\cos(2t)$. Since $y(0) = 3$ and $z(0) = 0$, we have $A = 3$ and $B = 0$. Thus the particle P travels counterclockwise around the circular path

$$\mathbf{r} = \mathbf{i} + 3\cos(2t)\mathbf{j} + 3\sin(2t)\mathbf{k}$$

in the plane $x = 1$ with angular speed 2. ■

Remark Newton's Second Law states that $\mathbf{F} = (d/dt)(m\mathbf{v}) = d\mathbf{p}/dt$, where $\mathbf{p} = m\mathbf{v}$ is the (linear) momentum of a particle of mass m moving under the influence of a force $\mathbf{F}$. This law may be reformulated in a manner appropriate for describing rotational motion as follows. If $\mathbf{r}(t)$ is the position of the particle at time t, then, since $\mathbf{v}\times\mathbf{v} = \mathbf{0}$,

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{p}) = \frac{d}{dt}\Big(\mathbf{r}\times(m\mathbf{v})\Big) = \mathbf{v}\times(m\mathbf{v}) + \mathbf{r}\times\frac{d}{dt}(m\mathbf{v}) = \mathbf{r}\times\mathbf{F}.$$

The quantities $\mathbf{H} = \mathbf{r}\times(m\mathbf{v})$ and $\mathbf{T} = \mathbf{r}\times\mathbf{F}$ are, respectively, the **angular momentum** of the particle about the origin and the **torque** of $\mathbf{F}$ about the origin. We have shown that

$$\mathbf{T} = \frac{d\mathbf{H}}{dt};$$

the torque of the external forces is equal to the rate of change of the angular momentum of the particle. This is the analogue for rotational motion of $\mathbf{F} = d\mathbf{p}/dt$.

Rotating Frames and the Coriolis Effect

The procedure of differentiating a vector function by differentiating its components is valid only if the basis vectors themselves do not depend on the variable of differentiation. In some situations in mechanics this is not the case. For instance, in modelling large-scale weather phenomena the analysis is affected by the fact that a coordinate system fixed with respect to the earth is, in fact, rotating (along with the earth) relative to directions fixed in space.

In order to understand the effect that the rotation of the coordinate system has on representations of velocity and acceleration, let us consider two Cartesian coordinate frames (i.e., systems of axes with corresponding unit basis vectors), a "fixed" frame with basis $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$, not rotating with the earth, and a rotating frame with basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ attached to the earth and therefore rotating with the same angular speed as the earth, namely, $\pi/12$ radians/hour. Let us take the origin of the fixed frame to be at the centre of the earth, with $\mathbf{K}$ pointing north. Then the angular velocity of the earth is $\boldsymbol{\Omega} = (\pi/12)\mathbf{K}$. The fixed frame is being carried along with the earth in its orbit around the sun, but it is not rotating with the earth, and, since the earth's orbital rotation around the sun has angular speed only 1/365th of the angular speed of its rotation about its axis, we can ignore the much smaller effect of the motion of the earth along its orbit.

Let us take the origin of the rotating frame to be at the location of an observer on the surface of the earth, say at point P_0 with position vector $\mathbf{R}_0$ with respect to the fixed frame.[1] Assume that P_0 has colatitude ϕ (the angle between $\mathbf{R}_0$ and $\mathbf{K}$) satisfying $0 < \phi < \pi$, so that P_0 is not at either the north pole or the south pole. Let us assume that $\mathbf{i}$ and $\mathbf{j}$ point, respectively, due east and north at P_0. Thus, $\mathbf{k}$ must point directly upward there. (See Figure 11.7 below.)

Since each of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and $\mathbf{R}_0$ is rotating with the earth (with angular velocity $\boldsymbol{\Omega}$), we have, as shown earlier in this section,

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\Omega}\times\mathbf{i}, \quad \frac{d\mathbf{j}}{dt} = \boldsymbol{\Omega}\times\mathbf{j}, \quad \frac{d\mathbf{k}}{dt} = \boldsymbol{\Omega}\times\mathbf{k}, \quad \text{and} \quad \frac{d\mathbf{R}_0}{dt} = \boldsymbol{\Omega}\times\mathbf{R}_0.$$

Any vector function can be expressed in terms of either basis. Let us denote by $\mathbf{R}(t)$, $\mathbf{V}(t)$, and $\mathbf{A}(t)$ the position, velocity, and acceleration of a moving object with respect to the fixed frame, and by $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ the same quantities with respect to the rotating frame. Thus,

$$\begin{aligned} \mathbf{R} &= X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K} & \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{V} &= \frac{dX}{dt}\mathbf{I} + \frac{dY}{dt}\mathbf{J} + \frac{dZ}{dt}\mathbf{K} & \mathbf{v} &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\ \mathbf{A} &= \frac{d^2X}{dt^2}\mathbf{I} + \frac{d^2Y}{dt^2}\mathbf{J} + \frac{d^2Z}{dt^2}\mathbf{K} & \mathbf{a} &= \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \end{aligned}$$

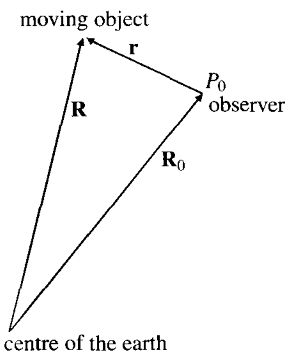

Figure 11.5 Position vectors relative to the fixed and rotating frames

How are the rotating-frame values of these vectors related to the fixed-frame values? Since the origin of the rotating frame is at $\mathbf{R}_0$, we have (see Figure 11.5)

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{r}.$$

When we differentiate with respect to time, we must remember that $\mathbf{R}_0$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ all depend on time. Therefore,

$$\begin{aligned} \mathbf{V} = \frac{d\mathbf{R}}{dt} &= \frac{d\mathbf{R}_0}{dt} + \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt} + \frac{dy}{dt}\mathbf{j} + y\frac{d\mathbf{j}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt} \\ &= \mathbf{v} + \boldsymbol{\Omega}\times\mathbf{R}_0 + x\boldsymbol{\Omega}\times\mathbf{i} + y\boldsymbol{\Omega}\times\mathbf{j} + z\boldsymbol{\Omega}\times\mathbf{k} \\ &= \mathbf{v} + \boldsymbol{\Omega}\times\mathbf{R}_0 + \boldsymbol{\Omega}\times\mathbf{r} \\ &= \mathbf{v} + \boldsymbol{\Omega}\times\mathbf{R}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{A} = \frac{d\mathbf{V}}{dt} &= \frac{d}{dt}(\mathbf{v} + \boldsymbol{\Omega}\times\mathbf{R}) \\ &= \frac{d^2x}{dt^2}\mathbf{i} + \frac{dx}{dt}\frac{d\mathbf{i}}{dt} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{dy}{dt}\frac{d\mathbf{j}}{dt} + \frac{d^2z}{dt^2}\mathbf{k} + \frac{dz}{dt}\frac{d\mathbf{k}}{dt} + \boldsymbol{\Omega}\times\frac{d\mathbf{R}}{dt} \\ &= \mathbf{a} + \boldsymbol{\Omega}\times\mathbf{v} + \boldsymbol{\Omega}\times(\mathbf{V}) \\ &= \mathbf{a} + 2\boldsymbol{\Omega}\times\mathbf{v} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R}). \end{aligned}$$

[1] The author is grateful to his colleague, Professor Lon Rosen, for suggesting this approach to the analysis of the rotating frame.

The term $2\boldsymbol{\Omega}\times\mathbf{v}$ is called the **Coriolis acceleration**, and the term $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R})$ is called the **centripetal acceleration**.

Suppose our moving object has mass m and is acted on by an external force $\mathbf{F}$. By Newton's Second Law,

$$\mathbf{F} = m\mathbf{A} = m\mathbf{a} + 2m\boldsymbol{\Omega}\times\mathbf{v} + m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R}),$$

or, equivalently,

$$\mathbf{a} = \frac{\mathbf{F}}{m} - 2\boldsymbol{\Omega}\times\mathbf{v} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R}).$$

To the observer on the rotating earth, the object appears to be subject to $\mathbf{F}$ and to two other forces, a **Coriolis force**, whose value per unit mass is $-2\boldsymbol{\Omega}\times\mathbf{v}$, and a **centrifugal force**, whose value per unit mass is $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R})$. The centrifugal and Coriolis forces are not "real" forces acting on the object. They are fictitious forces that compensate for the fact that we are measuring acceleration with respect to a frame that we are regarding as fixed, although it is really rotating and hence accelerating.

Observe that the centrifugal force points directly away from the polar axis of the earth. It represents the effect that the moving object wants to continue moving in a straight line and "fly off" from the earth rather than continuing to rotate along with the observer. This force is greatest at the equator (where $\boldsymbol{\Omega}$ is perpendicular to $\mathbf{R}$), but it is of very small magnitude: $|\boldsymbol{\Omega}|^2|\mathbf{R}_0| \approx 0.003\text{g}$.

The Coriolis force is quite different in nature from the centrifugal force. In particular, it is zero if the observer perceives the object to be at rest. It is perpendicular to both the velocity of the object and the polar axis of the earth, and its magnitude can be as large as $2|\boldsymbol{\Omega}||\mathbf{v}|$ and, in particular, can be larger than that of the centrifugal force if $|\mathbf{v}|$ is sufficiently large.

Example 3 **(Winds around the eye of a storm)** The circulation of winds around a storm centre is an example of the Coriolis effect. The eye of a storm is an area of low pressure sucking air toward it. The direction of rotation of the earth is such that the angular velocity $\boldsymbol{\Omega}$ points north and is parallel to the earth's axis of rotation. At any point P on the surface of the earth we can express $\boldsymbol{\Omega}$ as a sum of tangential (to the earth's surface) and normal components (see Figure 11.6(a)),

$$\boldsymbol{\Omega}(P) = \boldsymbol{\Omega}_T(P) + \boldsymbol{\Omega}_N(P).$$

If P is in the northern hemisphere, $\boldsymbol{\Omega}_N(P)$ points upward (away from the centre of the earth). At such a point the Coriolis "force" $\mathbf{C} = -2\boldsymbol{\Omega}(P)\times\mathbf{v}$ on a particle of air moving with horizontal velocity $\mathbf{v}$ would itself have horizontal and normal components

$$\mathbf{C} = -2\boldsymbol{\Omega}_T\times\mathbf{v} - 2\boldsymbol{\Omega}_N\times\mathbf{v} = \mathbf{C}_N + \mathbf{C}_T.$$

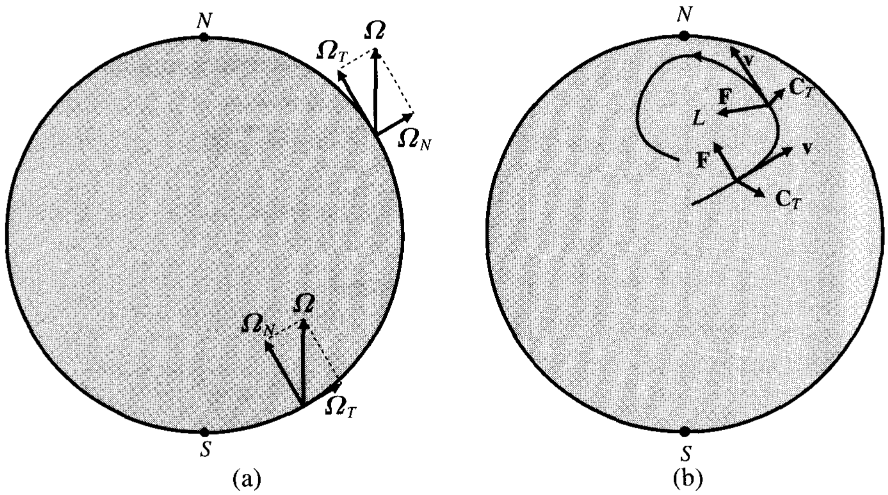

Figure 11.6

(a) Tangential and normal components of the angular velocity of the earth in the northern and southern hemispheres

(b) In the northern hemisphere the tangential Coriolis force deflects winds to the right of the path toward the low-pressure area L so the winds move counterclockwise around the centre of L

The normal component of the Coriolis force has negligible effect, since air is not free to travel great distances vertically. However, the tangential component of the Coriolis force, $\mathbf{C}_T = -2\boldsymbol{\Omega}_N \times \mathbf{v}$ is 90° to the right of $\mathbf{v}$ (i.e., clockwise from $\mathbf{v}$). Therefore, particles of air that are being sucked toward the eye of the storm experience Coriolis deflection to the right and so actually spiral into the eye in a counterclockwise direction. The opposite is true in the southern hemisphere, where the normal component $\boldsymbol{\Omega}_N$ is downward (into the earth). The suction force $\mathbf{F}$, the velocity $\mathbf{v}$, and the component of the Coriolis force tangential to the earth's surface, $\mathbf{C}_T$, are shown at two positions on the path of an air particle spiralling around a low-pressure area in the northern hemisphere in Figure 11.6(b). ■

Remark Strong winds spiralling inward around low-pressure areas are called **cyclones**. Strong winds spiralling outward around high-pressure areas are called **anticyclones**. The latter spiral counterclockwise in the southern hemisphere and clockwise in the northern hemisphere. The Coriolis effect also accounts for the high-velocity eastward-flowing jet streams in the upper atmosphere at midlatitudes in both hemispheres, the energy being supplied by the rising of warm tropical air and its subsequent moving toward the poles.

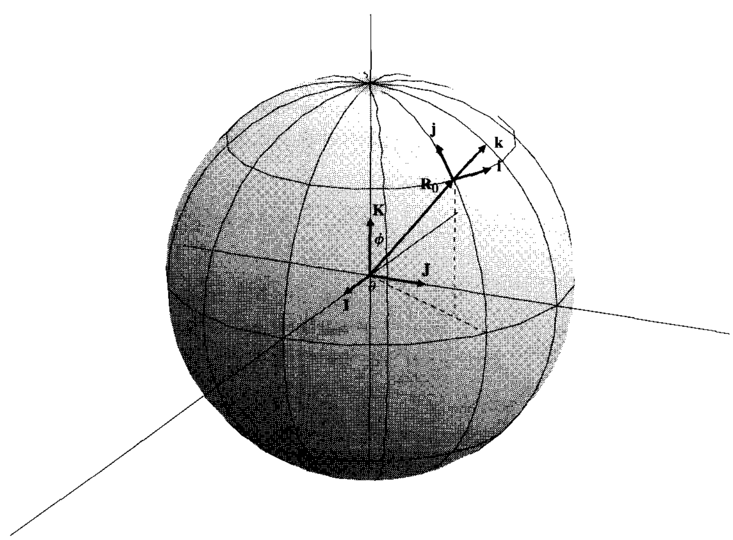

Figure 11.7 The fixed and local frames

The relationships between the basis vectors in the fixed and rotating frames can be used to analyze many phenomena. Recall that $\mathbf{R}_0$ makes angle ϕ with $\mathbf{K}$. Suppose the projection of $\mathbf{R}_0$ onto the equatorial plane (containing $\mathbf{I}$ and $\mathbf{J}$) makes angle θ with $\mathbf{I}$ as shown in Figure 11.7. Careful consideration of that figure should convince you that

$$\begin{aligned}\mathbf{i} &= -\sin\theta\mathbf{I} + \cos\theta\mathbf{J}\\ \mathbf{j} &= -\cos\phi\cos\theta\mathbf{I} - \cos\phi\sin\theta\mathbf{J} + \sin\phi\mathbf{K}\\ \mathbf{k} &= \sin\phi\cos\theta\mathbf{I} + \sin\phi\sin\theta\mathbf{J} + \cos\phi\mathbf{K}.\end{aligned}$$

Similarly, or by solving the above equations for $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$,

$$\begin{aligned}\mathbf{I} &= -\sin\theta\mathbf{i} - \cos\phi\cos\theta\mathbf{j} + \sin\phi\cos\theta\mathbf{k}\\ \mathbf{J} &= \cos\theta\mathbf{i} - \cos\phi\sin\theta\mathbf{j} + \sin\phi\sin\theta\mathbf{k}\\ \mathbf{K} &= \sin\phi\mathbf{j} + \cos\phi\mathbf{k}.\end{aligned}$$

Note that as the earth rotates on its axis, ϕ remains constant while θ increases at $(\pi/12)$ radians/hour.

Example 4 Suppose that the direction to the sun lies in the plane of $\mathbf{I}$ and $\mathbf{K}$, and makes angle σ with $\mathbf{I}$. Thus, the sun lies in the direction of the vector

$$\mathbf{S} = \cos\sigma\mathbf{I} + \sin\sigma\mathbf{K}.$$

($\sigma = 0$ at the March and September equinoxes and $\sigma \approx 23.3^\circ$ and -23.3° at the June and December solstices.) Find the length of the day (the time between sunrise and sunset) for an observer at colatitude ϕ.

Solution The sun will be "up" for the observer if the angle between $\mathbf{S}$ and $\mathbf{k}$ does not exceed $\pi/2$, that is, if $\mathbf{S} \bullet \mathbf{k} \ge 0$. Thus, daytime corresponds to

$$\cos\sigma\sin\phi\cos\theta + \sin\sigma\cos\phi \ge 0,$$

or, equivalently, $\cos\theta \ge -\dfrac{\tan\sigma}{\tan\phi}$. Sunup and sundown occur where equality occurs, namely, when

$$\theta = \theta_0 = \pm\cos^{-1}\left(-\frac{\tan\sigma}{\tan\phi}\right)$$

if such values exist. (They will exist if $\phi \ge \sigma \ge 0$ or if $\pi - \phi \ge -\sigma \ge 0$.) In this case, daytime for the observer lasts

$$\frac{2\theta_0}{2\pi} \times 24 = \frac{24}{\pi}\cos^{-1}\left(-\frac{\tan\sigma}{\tan\phi}\right) \text{ hours.}$$

For instance, on June 21st at the Arctic Circle (so $\phi = \sigma$), daytime lasts $(24/\pi)\cos^{-1}(-1) = 24$ hours. ■

Exercises 11.2

1. What fraction of its total initial mass would the rocket considered in Example 1 have to burn as fuel in order to accelerate in a straight line from rest to the speed of its own exhaust gases? to twice that speed?

* 2. When run at maximum power output, the motor in a self-propelled tank car can accelerate the full car (mass M kg) along a horizontal track at a m/s^2. The tank is full at time zero but the contents pour out of a hole in the bottom at rate k kg/s thereafter. If the car is at rest at time zero and full forward power is turned on at that time, how fast will it be moving at any time t before the tank is empty?

◆ 3. Solve the initial-value problem

$$\frac{d\mathbf{r}}{dt} = \mathbf{k}\times\mathbf{r}, \qquad \mathbf{r}(0) = \mathbf{i}+\mathbf{k}.$$

Describe the curve $\mathbf{r} = \mathbf{r}(t)$.

◆ 4. An object moves so that its position vector $\mathbf{r}(t)$ satisfies

$$\frac{d\mathbf{r}}{dt} = \mathbf{a}\times\big(\mathbf{r}(t) - \mathbf{b}\big)$$

and $\mathbf{r}(0) = \mathbf{r}_0$. Here, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{r}_0$ are given constant vectors with $\mathbf{a} \neq \mathbf{0}$. Describe the path along which the object moves.

The Coriolis effect

* 5. A satellite is in a low, circular, polar orbit around the earth, (i.e., passing over the north and south poles). It makes one revolution every two hours. An observer standing on the earth at the equator sees the satellite pass directly overhead. In what direction does it seem to the observer to be moving? From the observer's point of view, what is the approximate value of the Coriolis force acting on the satellite?

* 6. Repeat the previous exercise for an observer at a latitude of 45° in the northern hemisphere.

* 7. Describe the tangential and normal components of the Coriolis force on a particle moving with horizontal velocity $\mathbf{v}$ at (a) the north pole, (b) the south pole, and (c) the equator. In general, what is the effect of the normal component of the Coriolis force near the eye of a storm?

* 8. **(The location of sunrise and sunset)** Extend the argument in Example 4 to determine where on the horizon of the observer at P_0 the sun will rise and set. Specifically, if μ is the angle between $\mathbf{j}$ and $\mathbf{S}$ (the direction to the sun) at sunrise or sunset, show that

$$\cos\mu = \frac{\sin\sigma}{\sin\phi}.$$

For example, if $\sigma = 0$ (the equinoxes), then $\mu = \pi/2$ at all colatitudes ϕ; the sun rises due east and sets due west on those days.

9. Vancouver, Canada, has latitude 49.2° N, so its colatitude is 40.8°. How long is the sun visible at Vancouver on June 21st? Or rather, how long would it be visible if it weren't raining and if there were not so many mountains around? At what angle away from north would the sun rise and set?

10. Repeat the previous exercise for Umeå, Sweden (latitude 63.5° N).

11.3 Curves and Parametrizations

In this section we will consider curves as geometric objects rather than as paths of moving particles. Everyone has an intuitive idea of what a curve is, but it is difficult to give a formal definition of a curve as a geometric object (i.e., as a certain kind of set of points) without involving the concept of parametric representation. We will avoid this difficulty by continuing to regard a curve in 3-space as the set of points whose positions are given by the position vector function

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b.$$

However, the parameter t need no longer represent time, or any other specific physical quantity.

Curves can be very pathological. For instance, there exist continuous curves that pass through every point in a cube. It is difficult to think of such a curve as a one-dimensional object. In order to avoid such strange objects we *assume* hereafter that the defining function $\mathbf{r}(t)$ has a *continuous* first derivative, $d\mathbf{r}/dt$, which we will continue to call "velocity" and denote by $\mathbf{v}(t)$ by analogy with the physical case where t is time. (We also continue to call $v(t) = |\mathbf{v}(t)|$ the "speed.") As we will see later, this implies that the curve has an *arc length* between any two points corresponding to parameter values t_1 and t_2; if $t_1 < t_2$, this arc length is

$$\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} |\mathbf{v}(t)|\,dt = \int_{t_1}^{t_2} \left|\frac{d\mathbf{r}}{dt}\right| dt.$$

Frequently we will want $\mathbf{r}(t)$ to have continuous derivatives of higher order. Whenever needed, we will assume that the "acceleration", $\mathbf{a}(t) = d^2\mathbf{r}/dt^2$, and even the third derivative, $d^3\mathbf{r}/dt^3$, are continuous. Of course, most of the curves we encounter in practice have parametrizations with continuous derivatives of all orders.

It must be recalled, however, that no assumptions on the continuity of derivatives of the function $\mathbf{r}(t)$ are sufficient to guarantee that the curve $\mathbf{r} = \mathbf{r}(t)$ is a "smooth" curve. It may fail to be smooth at a point where $\mathbf{v} = \mathbf{0}$. (See Example 1 in Section 11.1.) We will show in the next section that if, besides being continuous, the velocity vector $\mathbf{v}(t)$ is *never the zero vector*, then the curve $\mathbf{r} = \mathbf{r}(t)$ is **smooth** in the sense that it has a continuously turning tangent line.

Although we have said that a curve is a set of points given by a parametric equation $\mathbf{r} = \mathbf{r}(t)$, there is no *unique* way of representing a given curve parametrically. Just as two cars can travel the same highway at different speeds, stopping and starting at different places, so too can the same curve be defined by different parametrizations; a given curve can have infinitely many different parametrizations.

Example 1 Show that each of the vector functions

$$\begin{aligned}
\mathbf{r}_1(t) &= \sin t\,\mathbf{i} + \cos t\,\mathbf{j}, && (-\pi/2 \le t \le \pi/2),\\
\mathbf{r}_2(t) &= (t-1)\,\mathbf{i} + \sqrt{2t - t^2}\,\mathbf{j}, && (0 \le t \le 2), \quad \text{and}\\
\mathbf{r}_3(t) &= t\sqrt{2 - t^2}\,\mathbf{i} + (1 - t^2)\,\mathbf{j}, && (-1 \le t \le 1)
\end{aligned}$$

all represent the same curve. Describe the curve.

Solution All three functions represent points in the xy-plane. The function $\mathbf{r}_1(t)$ starts at the point $(-1, 0)$ with position vector $\mathbf{r}_1(-\pi/2) = -\mathbf{i}$ and ends at the point $(1, 0)$ with position vector $\mathbf{i}$. It lies in the half of the xy-plane where $y \ge 0$ (because $\cos t \ge 0$ for $(-\pi/2 \le t \le \pi/2)$). Finally, all points on the curve are at distance 1 from the origin:

$$|\mathbf{r}_1(t)| = \sqrt{(\sin t)^2 + (\cos t)^2} = 1.$$

Therefore, $\mathbf{r}_1(t)$ represents the semicircle $y = \sqrt{1 - x^2}$ in the xy-plane traversed from left to right.

The other two functions have the same properties: both graphs lie in $y \ge 0$,

$$\begin{aligned}
\mathbf{r}_2(0) &= -\mathbf{i}, & \mathbf{r}_2(2) &= \mathbf{i}, & |\mathbf{r}_2(t)| &= \sqrt{(t-1)^2 + 2t - t^2} = 1,\\
\mathbf{r}_3(-1) &= -\mathbf{i}, & \mathbf{r}_3(1) &= \mathbf{i}, & |\mathbf{r}_3(t)| &= \sqrt{t^2(2 - t^2) + (1 - t^2)^2} = 1.
\end{aligned}$$

Thus all three functions represent the same semicircle (see Figure 11.8). Of course, the three parametrizations trace out the curve with different velocities. ■

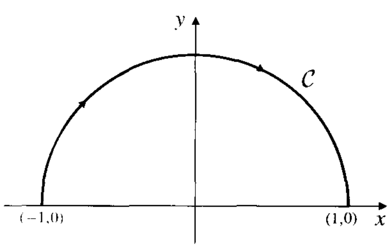

Figure 11.8 Three parametrizations of the semicircle C are given in Example 1

The curve $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$ is called a **closed curve** if $\mathbf{r}(a) = \mathbf{r}(b)$, that is, if the curve begins and ends at the same point. The curve $\mathcal{C}$ is **non-self-intersecting** if there exists some parametrization $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$, of $\mathcal{C}$ that is one-to-one except that the endpoints could be the same:

$$\mathbf{r}(t_1) = \mathbf{r}(t_2) \quad a \le t_1 < t_2 \le b \quad \Longrightarrow t_1 = a \quad \text{and} \quad t_2 = b.$$

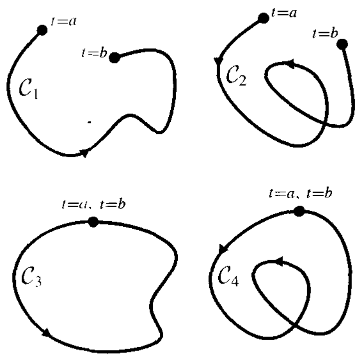

Figure 11.9 Curves $\mathcal{C}_1$ and $\mathcal{C}_3$ are non-self-intersecting
Curves $\mathcal{C}_2$ and $\mathcal{C}_4$ intersect themselves
Curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are not closed
Curves $\mathcal{C}_3$ and $\mathcal{C}_4$ are closed
Curve $\mathcal{C}_3$ is a simple closed curve

Such a curve can be closed, but otherwise does not intersect itself; it is then called a **simple closed curve.** Circles and ellipses are examples of simple closed curves. Every parametrization of a particular curve determines one of two possible **orientations** corresponding to the direction along the curve in which the parameter is increasing. Figure 11.9 illustrates these various concepts. All three parametrizations of the semicircle in the Example 1 orient the semicircle clockwise as viewed from a point above the xy-plane. This orientation is shown by the arrowheads on the curve in Figure 11.8. The same semicircle could be given the opposite orientation by, for example, the parametrization

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \qquad 0 \le t \le \pi.$$

Parametrizing the Curve of Intersection of Two Surfaces

Frequently a curve is specified as the intersection of two surfaces with given Cartesian equations. We may want to represent the curve by parametric equations. There is no unique way to do do this, but if one of the given surfaces is a cylinder parallel to a coordinate axis (so its equation is independent of one of the variables) we can begin by parametrizing that surface. The following examples clarify the method.

Example 2 Parametrize the curve of intersection of the plane $x + 2y + 4z = 4$ and the elliptic cylinder $x^2 + 4y^2 = 4$.

Solution We begin with the equation $x^2 + 4y^2 = 4$, which is independent of z. It can be parametrized in many ways; one convenient way is

$$x = 2\cos t, \qquad y = \sin t, \qquad (0 \le t \le 2\pi).$$

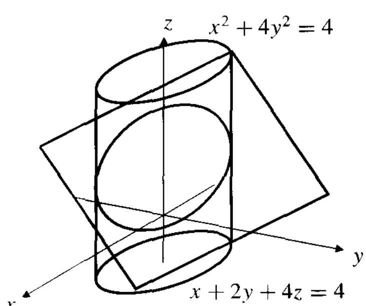

Figure 11.10 The curve of intersection of an oblique plane and an elliptic cylinder

The equation of the plane can then be solved for z, so that z can be expressed in terms of t:

$$z = \frac{1}{4}(4 - x - 2y) = 1 - \frac{1}{2}(\cos t + \sin t).$$

Thus, the given surfaces intersect in the curve (see Figure 11.10)

$$\mathbf{r} = 2\cos t\mathbf{i} + \sin t\mathbf{j} + \left(1 - \frac{\cos t + \sin t}{2}\right)\mathbf{k} \qquad (0 \le t \le 2\pi).$$

■

Example 3 Find a parametric representation of the curve of intersection of the two surfaces

$$x^2 + y + z = 2 \qquad \text{and} \qquad xy + z = 1.$$

Solution Here, neither given equation is independent of a variable, but we can obtain a third equation representing a surface containing the curve of intersection of the two given surfaces by subtracting the two given equations to eliminate z:

$$x^2 + y - xy = 1.$$

This equation is readily parametrized. If, for example, we let $x = t$, then

$$t^2 + y(1-t) = 1, \qquad \text{so} \quad y = \frac{1-t^2}{1-t} = 1 + t.$$

Either of the given equations can then be used to express z in terms of t:

$$z = 1 - xy = 1 - t(1+t) = 1 - t - t^2.$$

Thus, a possible parametrization of the curve is

$$\mathbf{r} = t\mathbf{i} + (1+t)\mathbf{j} + (1 - t - t^2)\mathbf{k}.$$

Of course, this answer is not unique. Many other parametrizations can be found for the curve, providing orientations in either direction.

Arc Length

We now consider how to define and calculate the length of a curve. Let $\mathcal{C}$ be a bounded, continuous curve specified by

$$\mathbf{r} = \mathbf{r}(t), \qquad a \le t \le b.$$

Subdivide the closed interval $[a, b]$ into n subintervals by points

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

The points $\mathbf{r}_i = \mathbf{r}(t_i)$, $(0 \le i \le n)$, subdivide $\mathcal{C}$ into n arcs. If we use the chord length $|\mathbf{r}_i - \mathbf{r}_{i-1}|$ as an approximation to the arc length between $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$, then the sum

$$s_n = \sum_{i=1}^{n} |\mathbf{r}_i - \mathbf{r}_{i-1}|$$

approximates the length of $\mathcal{C}$ by the length of a polygonal line. (See Figure 11.11.) Evidently, any such approximation is less than or equal to the actual length of $\mathcal{C}$. We say that $\mathcal{C}$ is **rectifiable** if there exists a constant K such that $s_n \le K$ for every n and every choice of the points t_i. In this case, the completeness axiom of the real number system assures us that there will be a smallest such number K. We call this smallest K the **length** of $\mathcal{C}$ and denote it by s.

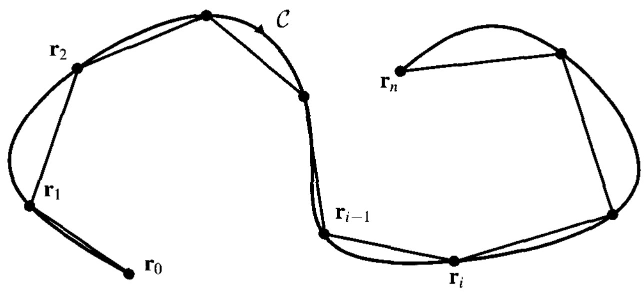

Figure 11.11 A polygonal approximation to a curve $\mathcal{C}$. The length of the polygonal line cannot exceed the length of the curve. In this figure the points on the curve are labelled with their position vectors, but the origin and these vectors are not themselves shown.

Let $\Delta t_i = t_i - t_{i-1}$ and $\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{r}_{i-1}$. Then s_n can be written in the form

$$s_n = \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{r}_i}{\Delta t_i} \right| \Delta t_i.$$

If $\mathbf{r}(t)$ has a continuous derivative $\mathbf{v}(t)$, then

$$s = \lim_{\max \Delta t_i \to 0} s_n = \int_a^b \left| \frac{d\mathbf{r}}{dt} \right| dt = \int_a^b |\mathbf{v}(t)|\, dt = \int_a^b v(t)\, dt.$$

In kinematic terms, this formula states that the distance travelled by a moving particle is the integral of the speed.

Remark Although the above formula is expressed in terms of the parameter t, the arc length, as defined above, is a strictly geometric property of the curve $\mathcal{C}$. It is independent of the particular parametrization used to represent $\mathcal{C}$. See Exercise 27 below.

If $s(t)$ denotes the arc length of that part of $\mathcal{C}$ corresponding to parameter values in $[a, t]$, then

$$\frac{ds}{dt} = \frac{d}{dt} \int_a^t v(\tau)\, d\tau = v(t),$$

so that the **arc length element** for $\mathcal{C}$ is given by

$$ds = v(t)\, dt = \left| \frac{d}{dt}\mathbf{r}(t) \right| dt.$$

The length of $\mathcal{C}$ is the integral of these arc length elements; we write

$$\int_{\mathcal{C}} ds = \text{length of } \mathcal{C} = \int_a^b v(t)\, dt.$$

Several familiar formulas for arc length follow from the above formula by using specific parametrizations of curves. For instance, the arc length element ds for the Cartesian plane curve $y = f(x)$ on $[a, b]$ is obtained by using x as parameter; here, $\mathbf{r} = x\mathbf{i} + f(x)\mathbf{j}$ so $\mathbf{v} = \mathbf{i} + f'(x)\mathbf{j}$ and

$$ds = \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

Similarly, the arc length element ds for a plane polar curve $r = g(\theta)$ can be calculated from the parametrization

$$\mathbf{r}(\theta) = g(\theta)\cos\theta\mathbf{i} + g(\theta)\sin\theta\mathbf{j}.$$

It is

$$ds = \sqrt{\left(g(\theta)\right)^2 + \left(g'(\theta)\right)^2}\,d\theta.$$

Example 4 Find the length s of that part of the **circular helix**

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$$

between the points $(a, 0, 0)$ and $(a, 0, 2\pi b)$.

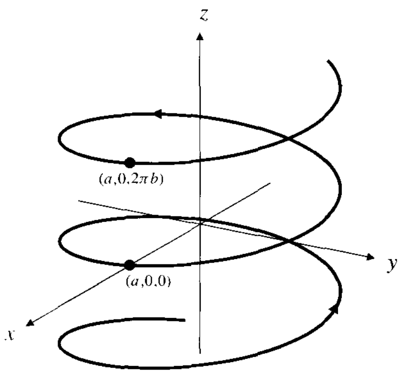

Figure 11.12 The helix
$x = a\cos t$
$y = a\sin t$
$z = bt$

Solution This curve spirals around the z-axis, rising as it turns. (See Figure 11.12.) It lies on the surface of the circular cylinder $x^2 + y^2 = a^2$. We have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\mathbf{k}$$
$$v = \sqrt{a^2 + b^2},$$

so that in terms of the parameter t the helix is traced out at constant speed. The required length s corresponds to parameter interval $[0, 2\pi]$. Thus,

$$s = \int_0^{2\pi} v(t)\,dt = \int_0^{2\pi} \sqrt{a^2 + b^2}\,dt = 2\pi\sqrt{a^2 + b^2}.$$

Piecewise Smooth Curves

As observed earlier, a parametric curve $\mathcal{C}$ given by $\mathbf{r} = \mathbf{r}(t)$ can fail to be smooth at points where $d\mathbf{r}/dt = \mathbf{0}$. If there are finitely many such points, we will say that the curve is piecewise smooth.

In general, a **piecewise smooth curve** $\mathcal{C}$ consists of a finite number of smooth arcs, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$, as shown in Figure 11.13.

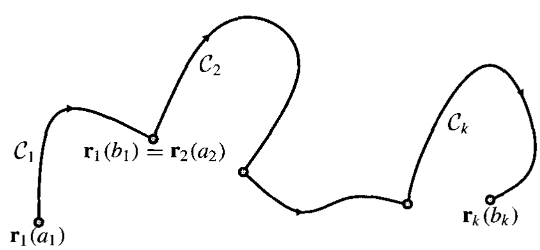

Figure 11.13 A piecewise smooth curve

In this case we express $\mathcal{C}$ as the sum of the individual arcs:

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_k.$$

Each arc $\mathcal{C}_i$ can have its own parametrization

$$\mathbf{r} = \mathbf{r}_i(t), \qquad (a_i \le t \le b_i),$$

where $\mathbf{v}_i = d\mathbf{r}_i/dt \neq \mathbf{0}$ for $a_i < t < b_i$. The fact that $\mathcal{C}_{i+1}$ must begin at the point where $\mathcal{C}_i$ ends requires the conditions

$$\mathbf{r}_{i+1}(a_{i+1}) = \mathbf{r}_i(b_i) \qquad \text{for } 1 \leq i \leq k-1.$$

If also $\mathbf{r}_k(b_k) = \mathbf{r}_1(a_1)$, then $\mathcal{C}$ is a closed piecewise smooth curve.

The length of a piecewise smooth curve $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_k$ is the sum of the lengths of its component arcs:

$$\text{length of } \mathcal{C} = \sum_{i=1}^{k} \int_{a_i}^{b_i} \left| \frac{d\mathbf{r}_i}{dt} \right| dt.$$

The Arc-Length Parametrization

The selection of a particular parameter in terms of which to specify a given curve will usually depend on the problem in which the curve arises; there is no one "right way" to parametrize a curve. However, there is one parameter that is "natural" in that it arises from the geometry (shape and size) of the curve itself and not from any particular coordinate system in which the equation of the curve is to be expressed. This parameter is the *arc length* measured from some particular point (the *initial point*) on the curve. The position vector of an arbitrary point P on the curve can be specified as a function of the arc length s along the curve from the initial point P_0 to P,

$$\mathbf{r} = \mathbf{r}(s).$$

This equation is called an **arc-length parametrization** or **intrinsic parametrization** of the curve. Since $ds = v(t)\,dt$ for any parametrization $\mathbf{r} = \mathbf{r}(t)$, for the arc-length parametrization we have $ds = v(s)\,ds$. Thus $v(s) = 1$, identically; *a curve parametrized in terms of arc length is traced at unit speed.* Although it is seldom easy (and usually not possible) to find $\mathbf{r}(s)$ explicitly when the curve is given in terms of some other parameter, smooth curves always have such parametrizations (see Exercise 28 below), and they will prove useful when we develop the fundamentals of the *differential geometry* for 3-space curves in the next section.

Suppose that a curve is specified in terms of an arbitrary parameter t. If the arc length over a parameter interval $[t_0, t]$,

$$s = s(t) = \int_{t_0}^{t} \left| \frac{d}{d\tau}\mathbf{r}(\tau) \right| d\tau,$$

can be evaluated explicitly, and if the equation $s = s(t)$ can be explicitly solved for t as a function of s ($t = t(s)$), then the curve can be reparametrized in terms of arc length by substituting for t in the original parametrization:

$$\mathbf{r} = \mathbf{r}(t(s)).$$

Example 5 Parametrize the circular helix

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$$

in terms of the arc length measured from the point $(a, 0, 0)$ in the direction of increasing t. (See Figure 11.12.)

Solution The initial point corresponds to $t = 0$. As shown in Example 4, we have $ds/dt = \sqrt{a^2 + b^2}$, so

$$s = s(t) = \int_0^t \sqrt{a^2 + b^2}\, d\tau = \sqrt{a^2 + b^2}\, t.$$

Therefore, $t = s/\sqrt{a^2 + b^2}$ and the arc-length parametrization is

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} + a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j} + \frac{bs}{\sqrt{a^2+b^2}}\mathbf{k}.$$

■

Exercises 11.3

In Exercises 1–4, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

1. In terms of the y-coordinate, oriented counterclockwise
2. In terms of the x-coordinate, oriented clockwise
3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise
4. In terms of arc length measured from $(0, a)$, oriented clockwise
5. The cylinders $z = x^2$ and $z = 4y^2$ intersect in two curves, one of which passes through the point $(2, -1, 4)$. Find a parametrization of that curve using $t = y$ as parameter.
6. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

In Exercises 7–10, parametrize the curve of intersection of the given surfaces. *Note*: the answers are not unique.

7. $x^2 + y^2 = 9$ and $z = x + y$
8. $z = \sqrt{1 - x^2 - y^2}$ and $x + y = 1$
9. $z = x^2 + y^2$ and $2x - 4y - z - 1 = 0$
10. $yz + x = 1$ and $xz - x = 1$
11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter: (a) $t = x$, (b) $t = y$, and (c) $t = z$. Which of these choices for t leads to a single parametrization that represents the whole parabola? What is that parametrization? What happens with the other two choices?

* 12. The plane $x + y + z = 1$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle $\mathcal{C}$. Find the centre $\mathbf{r}_0$ and radius r of $\mathcal{C}$. Also find two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ parallel to the plane of $\mathcal{C}$. (*Hint:* to be specific, show that $\hat{\mathbf{v}}_1 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is one such vector; then find a second that is perpendicular to $\hat{\mathbf{v}}_1$.) Use your results to construct a parametrization of $\mathcal{C}$.

13. Find the length of the curve $\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.
14. For what values of the parameter λ is the length $s(T)$ of the curve $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}$, $(0 \le t \le T)$ given by $s(T) = T + T^3$?
15. Express the length of the curve $\mathbf{r} = at^2\,\mathbf{i} + bt\,\mathbf{j} + c\ln t\,\mathbf{k}$, $(1 \le t \le T)$ as a definite integral. Evaluate the integral if $b^2 = 4ac$.
16. Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t\sin t, \qquad y = a\sin^2 t, \qquad z = bt.$$

What is the length of $\mathcal{C}$ between $t = 0$ and $t = T > 0$?

17. Find the length of the conical helix $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $(0 \le t \le 2\pi)$. Why is the curve called a conical helix?
18. Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.
19. Let $\mathcal{C}$ be the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = t$ between $t = 0$ and $t = 2\pi$. Find the length of $\mathcal{C}$.

20. Find the length of the piecewise smooth curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$, $(-1 \le t \le 2)$.

21. Describe the piecewise smooth curve $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}$, $(0 \le t \le 1)$, and $\mathbf{r}_2(t) = (1-t)\mathbf{i} + (1+t)\mathbf{j}$, $(0 \le t \le 1)$.

* **22.** A cable of length L and circular cross-section of radius a is wound around a cylindrical spool of radius b with no overlapping and with the adjacent windings touching one another. What length of the spool is covered by the cable?

In Exercises 23–26, reparametrize the given curve in the same orientation in terms of arc length measured from the point where $t = 0$.

23. $\mathbf{r} = At\mathbf{i} + Bt\mathbf{j} + Ct\mathbf{k}, \qquad (A^2 + B^2 + C^2 > 0)$

24. $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} - e^{-t}\mathbf{k}$

* **25.** $\mathbf{r} = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j} + b\cos 2t\,\mathbf{k}, \qquad (0 \le t \le \frac{\pi}{2})$

* **26.** $\mathbf{r} = 3t\cos t\,\mathbf{i} + 3t\sin t\,\mathbf{j} + 2\sqrt{2}t^{3/2}\mathbf{k}$

* **27.** Let $\mathbf{r} = \mathbf{r}_1(t)$, $(a \le t \le b)$, and $\mathbf{r} = \mathbf{r}_2(u)$, $(c \le u \le d)$, be two parametrizations of the same curve $\mathcal{C}$, each one-to-one on its domain and each giving $\mathcal{C}$ the same orientation (so that $\mathbf{r}_1(a) = \mathbf{r}_2(c)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(d)$). Then for each t in $[a, b]$ there is a unique $u = u(t)$ such that $\mathbf{r}_2(u(t)) = \mathbf{r}_1(t)$. Show that

$$\int_a^b \left|\frac{d}{dt}\mathbf{r}_1(t)\right| dt = \int_c^d \left|\frac{d}{du}\mathbf{r}_2(u)\right| du,$$

and thus that the length of $\mathcal{C}$ is independent of parametrization.

* **28.** If the curve $\mathbf{r} = \mathbf{r}(t)$ has continuous, nonvanishing velocity $\mathbf{v}(t)$ on the interval $[a, b]$, and if t_0 is some point in $[a, b]$, show that the function

$$s = g(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\, du$$

is an increasing function on $[a, b]$ and so has an inverse:

$$t = g^{-1}(s) \iff s = g(t).$$

Hence, show that the curve can be parametrized in terms of arc length measured from $\mathbf{r}(t_0)$.

11.4 Curvature, Torsion, and the Frenet Frame

In this section and the next we develop the fundamentals of differential geometry of curves in 3-space. We will introduce several new scalar and vector functions associated with a curve $\mathcal{C}$. The most important of these are the curvature and torsion of the curve, and a right-handed triad of mutually perpendicular unit vectors forming a basis at any point on the curve, and called the Frenet frame. The curvature measures the rate at which a curve is turning (away from its tangent line) at any point. The torsion measures the rate at which the curve is twisting (out of the plane in which it is turning) at any point.

The Unit Tangent Vector

The velocity vector $\mathbf{v}(t) = d\mathbf{r}/dt$ is tangent to the parametric curve $\mathbf{r} = \mathbf{r}(t)$ at the point $\mathbf{r}(t)$ and points in the direction of the orientation of the curve there. Since we are assuming that $\mathbf{v}(t) \ne \mathbf{0}$, we can find a **unit tangent vector**, $\hat{\mathbf{T}}(t)$, at $\mathbf{r}(t)$ by dividing $\mathbf{v}(t)$ by its length:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{d\mathbf{r}}{dt}\Big/\left|\frac{d\mathbf{r}}{dt}\right|.$$

Recall that a curve parametrized in terms of arc length, $\mathbf{r} = \mathbf{r}(s)$, is traced at unit speed; $v(s) = 1$. In terms of arc-length parametrization, the unit tangent vector is

$$\hat{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds}.$$

Example 1 Find the unit tangent vector, $\hat{\mathbf{T}}$, for the circular helix of Example 4 of Section 11.3, in terms of both t and the arc-length parameter s.

Solution In terms of t we have

$$\begin{aligned}
\mathbf{r} &= a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\mathbf{k}\\
\mathbf{v}(t) &= -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\mathbf{k}\\
v(t) &= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}\\
\hat{\mathbf{T}}(t) &= -\frac{a}{\sqrt{a^2+b^2}}\sin t\,\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos t\,\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}.
\end{aligned}$$

In terms of the arc-length parameter (see Example 5 of Section 11.3)

$$\begin{aligned}
\mathbf{r}(s) &= a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} + a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j} + \frac{bs}{\sqrt{a^2+b^2}}\mathbf{k}\\
\hat{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds} &= -\frac{a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j}\\
&\quad + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}.
\end{aligned}$$

■

Remark If the curve $\mathbf{r} = \mathbf{r}(t)$ has a continuous, nonvanishing velocity $\mathbf{v}(t)$, then the unit tangent vector $\hat{\mathbf{T}}(t)$ is a continuous function of t. The angle $\theta(t)$ between $\hat{\mathbf{T}}(t)$ and any fixed unit vector $\hat{\mathbf{u}}$ is also continuous in t:

$$\theta(t) = \cos^{-1}(\hat{\mathbf{T}}(t) \bullet \hat{\mathbf{u}}).$$

Thus, as asserted previously, the curve is *smooth* in the sense that it has a continuously turning tangent line. The rate of this turning is quantified by the curvature, which we introduce now.

Curvature and the Unit Normal

In the rest of this section we will deal abstractly with a curve C parametrized in terms of arc length measured from some point on it:

$$\mathbf{r} = \mathbf{r}(s).$$

In the next section we return to curves with arbitrary parametrizations and apply the principles developed in this section to specific problems. Throughout we assume that the parametric equations of curves have continuous derivatives up to third order on the intervals where they are defined.

Having unit length, the tangent vector $\hat{\mathbf{T}}(s) = d\mathbf{r}/ds$ satisfies $\hat{\mathbf{T}}(s) \bullet \hat{\mathbf{T}}(s) = 1$. Differentiating this equation with respect to s we get

$$2\hat{\mathbf{T}}(s) \bullet \frac{d\hat{\mathbf{T}}}{ds} = 0,$$

so that $d\hat{\mathbf{T}}/ds$ is perpendicular to $\hat{\mathbf{T}}(s)$.

DEFINITION 1

Curvature and radius of curvature

The **curvature** of $\mathcal{C}$ at the point $\mathbf{r}(s)$ is the length of $d\hat{\mathbf{T}}/ds$ there. It is denoted by the Greek letter κ (*kappa*):

$$\kappa(s) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right|.$$

The **radius of curvature**, denoted ρ (the Greek letter *rho*) is the reciprocal of the curvature:

$$\rho(s) = \frac{1}{\kappa(s)}.$$

As we will see below, the curvature of $\mathcal{C}$ at $\mathbf{r}(s)$ measures the rate of turning of the tangent line to the curve there. The radius of curvature is the radius of the circle through $\mathbf{r}(s)$ that most closely approximates the curve $\mathcal{C}$ near that point.

According to its definition, $\kappa(s) \geq 0$ everywhere on $\mathcal{C}$. If $\kappa(s) \neq 0$ we can divide $d\hat{\mathbf{T}}/ds$ by its length, $\kappa(s)$, and obtain a unit vector $\hat{\mathbf{N}}(s)$ in the same direction. This unit vector is called the **unit principal normal** to $\mathcal{C}$ at $\mathbf{r}(s)$, or, more commonly, just the **unit normal**:

$$\hat{\mathbf{N}}(s) = \frac{1}{\kappa(s)}\frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{ds}\bigg/\left|\frac{d\hat{\mathbf{T}}}{ds}\right|.$$

Note that $\hat{\mathbf{N}}(s)$ is perpendicular to $\mathcal{C}$ at $\mathbf{r}(s)$ and points in the direction that $\hat{\mathbf{T}}$, and therefore $\mathcal{C}$, is turning. The principal normal is not defined at points where the curvature $\kappa(s)$ is zero. For instance, a straight line has no principal normal. Figure 11.14(a) shows $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ at a point on a typical curve.

Example 2 Let $a > 0$. Show that the curve $\mathcal{C}$ given by

$$\mathbf{r} = a\cos\left(\frac{s}{a}\right)\mathbf{i} + a\sin\left(\frac{s}{a}\right)\mathbf{j}$$

is a circle in the xy-plane having radius a and centre at the origin and that it is parametrized in terms of arc length. Find the curvature, the radius of curvature, and the unit tangent and principal normal vectors at any point on $\mathcal{C}$.

Solution Since

$$|\mathbf{r}(s)| = a\sqrt{\left(\cos\left(\frac{s}{a}\right)\right)^2 + \left(\sin\left(\frac{s}{a}\right)\right)^2} = a,$$

$\mathcal{C}$ is indeed a circle of radius a centred at the origin in the xy-plane. Since the speed

$$\left|\frac{d\mathbf{r}}{ds}\right| = \left|-\sin\left(\frac{s}{a}\right)\mathbf{i} + \cos\left(\frac{s}{a}\right)\mathbf{j}\right| = 1,$$

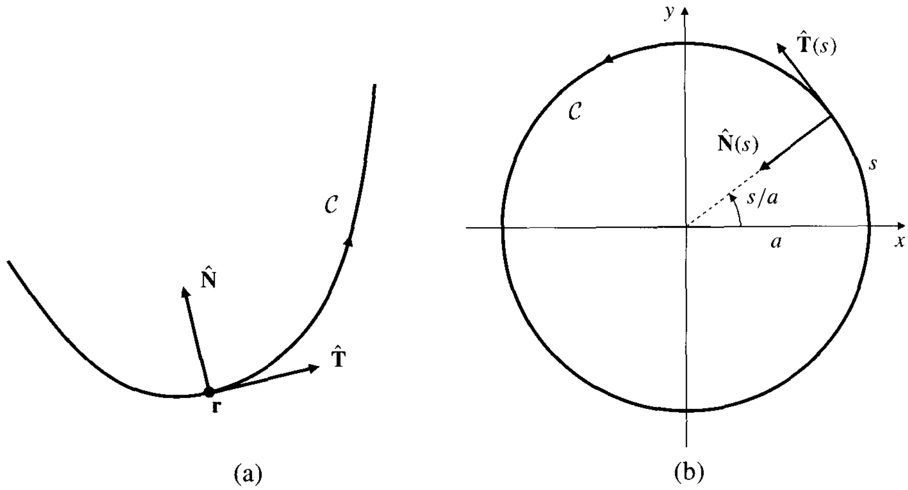

Figure 11.14

(a) The unit tangent and principal normal vectors for a curve

(b) The unit tangent and principal normal vectors for a circle

the parameter s must represent arc length; hence the unit tangent vector is

$$\hat{\mathbf{T}}(s) = -\sin\left(\frac{s}{a}\right)\mathbf{i} + \cos\left(\frac{s}{a}\right)\mathbf{j}.$$

Therefore,

$$\frac{d\hat{\mathbf{T}}}{ds} = -\frac{1}{a}\cos\left(\frac{s}{a}\right)\mathbf{i} - \frac{1}{a}\sin\left(\frac{s}{a}\right)\mathbf{j}$$

and the curvature and radius of curvature at $\mathbf{r}(s)$ are

$$\kappa(s) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{1}{a}, \qquad \rho(s) = \frac{1}{\kappa(s)} = a.$$

Finally, the unit principal normal is

$$\hat{\mathbf{N}}(s) = -\cos\left(\frac{s}{a}\right)\mathbf{i} - \sin\left(\frac{s}{a}\right)\mathbf{j} = -\frac{1}{a}\mathbf{r}(s).$$

Note that the curvature and radius of curvature are constant; the latter is in fact the radius of the circle. The circle and its unit tangent and normal vectors at a typical point are sketched in Figure 11.14(b). Note that $\hat{\mathbf{N}}$ points toward the centre of the circle.

■

Remark Another observation can be made about the above example. The position vector $\mathbf{r}(s)$ makes angle $\theta = s/a$ with the positive x-axis; therefore, $\hat{\mathbf{T}}(s)$ makes the same angle with the positive y-axis. Therefore, the rate of rotation of $\hat{\mathbf{T}}$ with respect to s is

$$\frac{d\theta}{ds} = \frac{1}{a} = \kappa.$$

That is, κ is the rate at which $\hat{\mathbf{T}}$ is turning (measured with respect to arc length). This observation extends to a general smooth curve.

THEOREM 2

Curvature is the rate of turning of the unit tangent

Let $\kappa > 0$ on an interval containing s, and let $\Delta\theta$ be the angle between $\hat{\mathbf{T}}(s + \Delta s)$ and $\hat{\mathbf{T}}(s)$, the unit tangent vectors at neighbouring points on the curve. Then

$$\kappa(s) = \lim_{\Delta s \to 0} \left| \frac{\Delta\theta}{\Delta s} \right|.$$

Figure 11.15 $|\Delta\hat{\mathbf{T}}| \approx |\Delta\theta|$ for small values of $|\Delta s|$

$$\lim_{\Delta s \to 0} \left| \frac{}{\Delta\theta} \right| = 1 \quad \text{and}$$

$$\kappa(s) = \lim_{\Delta s \to 0} \left| \frac{\Delta\hat{\mathbf{T}}}{\Delta s} \right| = \lim_{\Delta s \to 0} \left| \frac{\Delta\hat{\mathbf{T}}}{\Delta\theta} \right| \left| \frac{\Delta\theta}{\Delta s} \right| = \lim_{\Delta s \to 0} \left| \frac{\Delta\theta}{\Delta s} \right|.$$

The unit tangent $\hat{\mathbf{T}}$ and unit normal $\hat{\mathbf{N}}$ at a point $\mathbf{r}(s)$ on a curve $\mathcal{C}$ are regarded as having their tails at that point. They are perpendicular, and $\hat{\mathbf{N}}$ points in the direction toward which $\hat{\mathbf{T}}(s)$ turns as s increases. The plane passing through $\mathbf{r}(s)$ and containing the vectors $\hat{\mathbf{T}}(s)$ and $\hat{\mathbf{N}}(s)$ is called the **osculating plane** of $\mathcal{C}$ at $\mathbf{r}(s)$ (from the Latin *osculum*, meaning *kiss*). For a *plane curve*, such as the circle in Example 2, the osculating plane is just the plane containing the curve. For more general three-dimensional curves the osculating plane varies from point to point; at any point it is that plane which comes closest to containing the part of the curve near that point. The osculating plane is not properly defined at a point where $\kappa(s) = 0$, although if such points are isolated, it can sometimes be defined as a limit of osculating planes for neighbouring points.

Still assuming that $\kappa(s) \neq 0$, let

$$\mathbf{r}_c(s) = \mathbf{r}(s) + \rho(s)\hat{\mathbf{N}}(s).$$

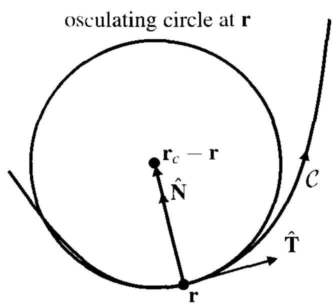

Figure 11.16 An osculating circle

For each s the point with position vector $\mathbf{r}_c(s)$ lies in the osculating plane of $\mathcal{C}$ at $\mathbf{r}(s)$, on the concave side of $\mathcal{C}$ and at distance $\rho(s)$ from $\mathbf{r}(s)$. It is called the **centre of curvature** of $\mathcal{C}$ for the point $\mathbf{r}(s)$. The circle in the osculating plane having centre at the centre of curvature and radius equal to the radius of curvature $\rho(s)$ is called the **osculating circle** for $\mathcal{C}$ at $\mathbf{r}(s)$. Among all circles that pass through the point $\mathbf{r}(s)$, the osculating circle is the one that best describes the behaviour of $\mathcal{C}$ near that point. Of course, the osculating circle of a circle at any point is the same circle. A typical example of an osculating circle is shown in Figure 11.16.

Torsion and Binormal, the Frenet–Serret Formulas

At any point $\mathbf{r}(s)$ on the curve $\mathcal{C}$ where $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are defined, a third unit vector, the **unit binormal $\hat{\mathbf{B}}$**, is defined by the formula

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}.$$

Note that $\hat{\mathbf{B}}(s)$ is normal to the osculating plane of $\mathcal{C}$ at $\mathbf{r}(s)$; if $\mathcal{C}$ is a plane curve, then $\hat{\mathbf{B}}$ is a constant vector, independent of s on any interval where $\kappa(s) \neq 0$. At each point $\mathbf{r}(s)$ on $\mathcal{C}$, the three vectors $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}\}$ constitute a right-handed basis of mutually perpendicular unit vectors like the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. (See Figure 11.17.) This basis is called the **Frenet frame** for $\mathcal{C}$ at the point $\mathbf{r}(s)$. Note that

$$\hat{\mathbf{B}} \times \hat{\mathbf{T}} = \hat{\mathbf{N}} \text{ and } \hat{\mathbf{N}} \times \hat{\mathbf{B}} = \hat{\mathbf{T}}.$$

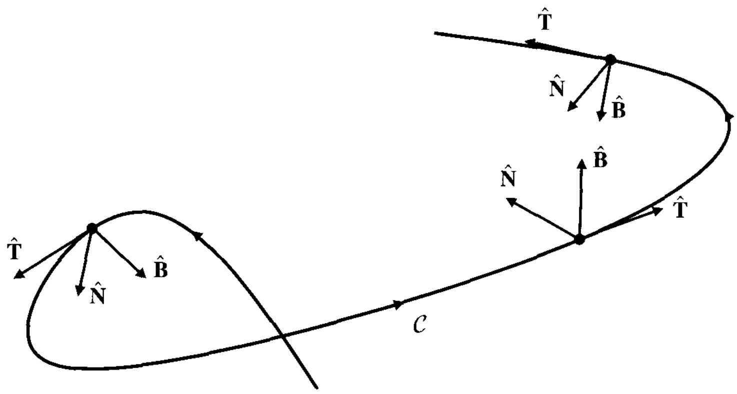

Figure 11.17 The Frenet frame $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}\}$ at some points on $\mathcal{C}$

Since $1 = \hat{\mathbf{B}}(s) \bullet \hat{\mathbf{B}}(s)$, then $\hat{\mathbf{B}}(s) \bullet (d\hat{\mathbf{B}}/ds) = 0$, and $d\hat{\mathbf{B}}/ds$ is perpendicular to $\hat{\mathbf{B}}(s)$. Also, differentiating $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$ we obtain

$$\frac{d\hat{\mathbf{B}}}{ds} = \frac{d\hat{\mathbf{T}}}{ds} \times \hat{\mathbf{N}} + \hat{\mathbf{T}} \times \frac{d\hat{\mathbf{N}}}{ds} = \kappa \hat{\mathbf{N}} \times \hat{\mathbf{N}} + \hat{\mathbf{T}} \times \frac{d\hat{\mathbf{N}}}{ds} = \hat{\mathbf{T}} \times \frac{d\hat{\mathbf{N}}}{ds}.$$

Therefore $d\hat{\mathbf{B}}/ds$ is also perpendicular to $\hat{\mathbf{T}}$. Being perpendicular to both $\hat{\mathbf{T}}$ and $\hat{\mathbf{B}}$, $d\hat{\mathbf{B}}/ds$ must be parallel to $\hat{\mathbf{N}}$. This fact is the basis for our definition of torsion.

DEFINITION 2

Torsion

On any interval where $\kappa(s) \neq 0$ there exists a function $\tau(s)$ such that

$$\frac{d\hat{\mathbf{B}}}{ds} = -\tau(s)\hat{\mathbf{N}}(s).$$

The number $\tau(s)$ is called the **torsion** of $\mathcal{C}$ at $\mathbf{r}(s)$.

The torsion measures the degree of twisting that the curve exhibits near a point, that is, the extent to which the curve fails to be planar. It may be positive or negative, depending on the right-handedness or left-handedness of the twisting. We will present an example later in this section.

Theorem 2 has an analogue for torsion, for which the proof is similar. It states that the absolute value of the torsion, $|\tau(s)|$, at point $\mathbf{r}(s)$ on the curve $\mathcal{C}$ is the rate of turning of the unit binormal:

$$\lim_{\Delta s \to 0} \left| \frac{\Delta\psi}{\Delta s} \right| = |\tau(s)|,$$

where $\Delta\psi$ is the angle between $\hat{\mathbf{B}}(s + \Delta s)$ and $\hat{\mathbf{B}}(s)$.

Example 3 **(The circular helix)** As observed in Example 5 of Section 11.3, the parametric equation

$$\mathbf{r}(s) = a\cos(cs)\mathbf{i} + a\sin(cs)\mathbf{j} + bcs\mathbf{k}, \qquad \text{where } c = \frac{1}{\sqrt{a^2+b^2}},$$

represents a circular helix wound on the cylinder $x^2 + y^2 = a^2$ and parametrized in terms of arc length. Assume $a > 0$. Find the curvature and torsion functions $\kappa(s)$ and $\tau(s)$ for this helix and also the unit vectors comprising the Frenet frame at any point $\mathbf{r}(s)$ on the helix.

Solution In Example 1 we calculated the unit tangent vector to be

$$\hat{\mathbf{T}}(s) = -ac\sin(cs)\mathbf{i} + ac\cos(cs)\mathbf{j} + bc\mathbf{k}.$$

Differentiating again leads to

$$\frac{d\hat{\mathbf{T}}}{ds} = -ac^2\cos(cs)\mathbf{i} - ac^2\sin(cs)\mathbf{j},$$

so that the curvature of the helix is

$$\kappa(s) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = ac^2 = \frac{a}{a^2+b^2},$$

and the unit normal vector is

$$\hat{\mathbf{N}}(s) = \frac{1}{\kappa(s)}\frac{d\hat{\mathbf{T}}}{ds} = -\cos(cs)\mathbf{i} - \sin(cs)\mathbf{j}.$$

Now we have

$$\begin{aligned}\hat{\mathbf{B}}(s) = \hat{\mathbf{T}}(s)\times\hat{\mathbf{N}}(s) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -ac\sin(cs) & ac\cos(cs) & bc \\ -\cos(cs) & -\sin(cs) & 0 \end{vmatrix} \\ &= bc\sin(cs)\mathbf{i} - bc\cos(cs)\mathbf{j} + ac\mathbf{k}.\end{aligned}$$

Differentiating this formula leads to

$$\frac{d\hat{\mathbf{B}}}{ds} = bc^2\cos(cs)\mathbf{i} + bc^2\sin(cs)\mathbf{j} = -bc^2\hat{\mathbf{N}}(s).$$

Therefore, the torsion is given by

$$\tau(s) = -(-bc^2) = \frac{b}{a^2+b^2}.$$

■

Remark Observe that the curvature $\kappa(s)$ and the torsion $\tau(s)$ are both constant (i.e., independent of s) for a circular helix. In the above example, $\tau > 0$ (assuming that $b > 0$). This corresponds to the fact that the helix is *right-handed*. (See

Figure 11.12 in the previous section.) If you grasp the helix with your right hand so your fingers surround it in the direction of increasing s (counterclockwise, looking down from the positive z-axis), then your thumb also points in the axial direction corresponding to increasing s (the upward direction). Had we started with a left-handed helix, such as

$$\mathbf{r} = a \sin t\, \mathbf{i} + a \cos t\, \mathbf{j} + bt\mathbf{k}, \qquad (a, b > 0),$$

we would have obtained $\tau = -b/(a^2 + b^2)$.

Making use of the formulas $d\hat{\mathbf{T}}/ds = \kappa\hat{\mathbf{N}}$ and $d\hat{\mathbf{B}}/ds = -\tau\hat{\mathbf{N}}$ we can calculate $d\hat{\mathbf{N}}/ds$ as well:

$$\begin{aligned}\frac{d\hat{\mathbf{N}}}{ds} = \frac{d}{ds}(\hat{\mathbf{B}}\times\hat{\mathbf{T}}) &= \frac{d\hat{\mathbf{B}}}{ds}\times\hat{\mathbf{T}} + \hat{\mathbf{B}}\times\frac{d\hat{\mathbf{T}}}{ds} \\ &= -\tau\hat{\mathbf{N}}\times\hat{\mathbf{T}} + \kappa\hat{\mathbf{B}}\times\hat{\mathbf{N}} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}.\end{aligned}$$

Together, the three formulas

$$\frac{d\hat{\mathbf{T}}}{ds} = \kappa\hat{\mathbf{N}}$$
$$\frac{d\hat{\mathbf{N}}}{ds} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}$$
$$\frac{d\hat{\mathbf{B}}}{ds} = -\tau\hat{\mathbf{N}}$$

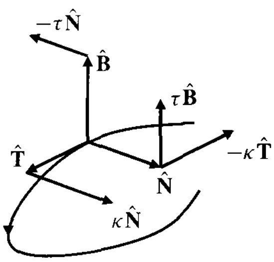

Figure 11.18 $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$, and their directions of change

are known as the **Frenet–Serret formulas**. (See Figure 11.18.) They are of fundamental importance in the theory of curves in 3-space. The Frenet–Serret formulas can be written in matrix form as follows:

$$\frac{d}{ds}\begin{pmatrix}\hat{\mathbf{T}}\\ \hat{\mathbf{N}}\\ \hat{\mathbf{B}}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\hat{\mathbf{T}}\\ \hat{\mathbf{N}}\\ \hat{\mathbf{B}}\end{pmatrix}.$$

Using the Frenet–Serret formulas we can show that the shape of a curve with non-vanishing curvature is completely determined by the curvature and torsion functions $\kappa(s)$ and $\tau(s)$.

THEOREM 3

The Fundamental Theorem of Space Curves

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two curves, both of which have the same nonvanishing curvature function $\kappa(s)$ and the same torsion function $\tau(s)$. Then the curves are congruent. That is, one can be moved rigidly (translated and rotated) so as to coincide exactly with the other.

PROOF We require $\kappa \neq 0$ because $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ are not defined where $\kappa = 0$. Move $\mathcal{C}_2$ rigidly so that its initial point coincides with the initial point of $\mathcal{C}_1$ and so that the Frenet frames of both curves coincide at that point. Let $\hat{\mathbf{T}}_1$, $\hat{\mathbf{T}}_2$, $\hat{\mathbf{N}}_1$, $\hat{\mathbf{N}}_2$, $\hat{\mathbf{B}}_1$, and $\hat{\mathbf{B}}_2$ be the unit tangents, normals, and binormals for the two curves. Let

$$f(s) = \hat{\mathbf{T}}_1(s) \bullet \hat{\mathbf{T}}_2(s) + \hat{\mathbf{N}}_1(s) \bullet \hat{\mathbf{N}}_2(s) + \hat{\mathbf{B}}_1(s) \bullet \hat{\mathbf{B}}_2(s).$$

We calculate the derivative of $f(s)$ using the Product Rule and the Frenet–Serret formulas:

$$\begin{aligned} f'(s) &= \hat{\mathbf{T}}_1' \bullet \hat{\mathbf{T}}_2 + \hat{\mathbf{T}}_1 \bullet \hat{\mathbf{T}}_2' + \hat{\mathbf{N}}_1' \bullet \hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_1 \bullet \hat{\mathbf{N}}_2' + \hat{\mathbf{B}}_1' \bullet \hat{\mathbf{B}}_2 + \hat{\mathbf{B}}_1 \bullet \hat{\mathbf{B}}_2' \\ &= \kappa \hat{\mathbf{N}}_1 \bullet \hat{\mathbf{T}}_2 + \kappa \hat{\mathbf{T}}_1 \bullet \hat{\mathbf{N}}_2 - \kappa \hat{\mathbf{T}}_1 \bullet \hat{\mathbf{N}}_2 + \tau \hat{\mathbf{B}}_1 \bullet \hat{\mathbf{N}}_2 - \kappa \hat{\mathbf{N}}_1 \bullet \hat{\mathbf{T}}_2 \\ &\qquad + \tau \hat{\mathbf{N}}_1 \bullet \hat{\mathbf{B}}_2 - \tau \hat{\mathbf{N}}_1 \bullet \hat{\mathbf{B}}_2 - \tau \hat{\mathbf{B}}_1 \bullet \hat{\mathbf{N}}_2 \\ &= 0. \end{aligned}$$

Therefore $f(s)$ is constant. Since the frames coincide at $s = 0$, the constant must be 3:

$$\hat{\mathbf{T}}_1(s) \bullet \hat{\mathbf{T}}_2(s) + \hat{\mathbf{N}}_1(s) \bullet \hat{\mathbf{N}}_2(s) + \hat{\mathbf{B}}_1(s) \bullet \hat{\mathbf{B}}_2(s) = 3.$$

However, each dot product cannot exceed 1 since the factors are unit vectors. Therefore each dot product must be equal to 1. In particular, $\hat{\mathbf{T}}_1(s) \bullet \hat{\mathbf{T}}_2(s) = 1$ for all s; hence

$$\frac{d\mathbf{r}_1}{ds} = \hat{\mathbf{T}}_1(s) = \hat{\mathbf{T}}_2(s) = \frac{d\mathbf{r}_2}{ds}.$$

Integrating with respect to s and using the fact that both curves start from the same point when $s = 0$, we obtain $\mathbf{r}_1(s) = \mathbf{r}_2(s)$ for all s, which is what we wanted to show.

Remark It is a consequence of the above theorem that any curve having nonzero constant curvature and constant torsion must, in fact, be a circle (if the torsion is zero) or a circular helix (if the torsion is nonzero). See Exercises 7 and 8 below.

Exercises 11.4

Find the unit tangent vector $\hat{\mathbf{T}}(t)$ for the curves in Exercises 1–4.

1. $\mathbf{r} = t\mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}$
2. $\mathbf{r} = a \sin \omega t\, \mathbf{i} + a \cos \omega t\, \mathbf{k}$
3. $\mathbf{r} = \cos t \sin t\, \mathbf{i} + \sin^2 t\, \mathbf{j} + \cos t\, \mathbf{k}$
4. $\mathbf{r} = a \cos t\, \mathbf{i} + b \sin t\, \mathbf{j} + t\mathbf{k}$
5. Show that if $\kappa(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a straight line.
6. Show that if $\tau(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a plane curve. *Hint:* show that $\mathbf{r}(s)$ lies in the plane through $\mathbf{r}(0)$ with normal $\hat{\mathbf{B}}(0)$.
7. Show that if $\kappa(s) = C$ is a positive constant and $\tau(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circle. *Hint:* find a circle having the given constant curvature. Then use Theorem 3.
8. Show that if the curvature $\kappa(s)$ and the torsion $\tau(s)$ are both nonzero constants, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circular helix. *Hint:* find a helix having the given curvature and torsion.

11.5 Curvature and Torsion for General Parametrizations

The formulas developed above for curvature and torsion as well as for the unit normal and binormal vectors are not very useful if the curve we want to analyze is not expressed in terms of the arc-length parameter. We will now consider how to find these quantities in terms of a general parametrization $\mathbf{r} = \mathbf{r}(t)$. We will express them all in terms of the velocity, $\mathbf{v}(t)$, the speed, $v(t) = |\mathbf{v}(t)|$, and the acceleration,

$\mathbf{a}(t)$. First observe that

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\hat{\mathbf{T}} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\hat{\mathbf{T}} + v\frac{d\hat{\mathbf{T}}}{dt} \\
&= \frac{dv}{dt}\hat{\mathbf{T}} + v\frac{d\hat{\mathbf{T}}}{ds}\frac{ds}{dt} = \frac{dv}{dt}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}} \\
\mathbf{v}\times\mathbf{a} &= v\frac{dv}{dt}\hat{\mathbf{T}}\times\hat{\mathbf{T}} + v^3\kappa\hat{\mathbf{T}}\times\hat{\mathbf{N}} = v^3\kappa\hat{\mathbf{B}}.
\end{aligned}$$

Note that $\hat{\mathbf{B}}$ is in the direction of $\mathbf{v}\times\mathbf{a}$. From these formulas we obtain useful formulas for $\hat{\mathbf{T}}$, $\hat{\mathbf{B}}$, and κ:

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{v}, \qquad \hat{\mathbf{B}} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}, \qquad \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3}.$$

There are several ways to calculate $\hat{\mathbf{N}}$. Perhaps the easiest is

$$\hat{\mathbf{N}} = \hat{\mathbf{B}}\times\hat{\mathbf{T}}.$$

Sometimes it may be easier to use $\dfrac{d\hat{\mathbf{T}}}{dt} = \dfrac{d\hat{\mathbf{T}}}{ds}\dfrac{ds}{dt} = v\dfrac{d\hat{\mathbf{T}}}{ds} = v\kappa\hat{\mathbf{N}}$ to calculate

$$\hat{\mathbf{N}} = \frac{1}{v\kappa}\frac{d\hat{\mathbf{T}}}{dt} = \frac{\rho}{v}\frac{d\hat{\mathbf{T}}}{dt} = \frac{d\hat{\mathbf{T}}}{dt}\bigg/\left|\frac{d\hat{\mathbf{T}}}{dt}\right|.$$

The torsion remains to be calculated. Observe that

$$\frac{d\mathbf{a}}{dt} = \frac{d}{dt}\left(\frac{dv}{dt}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}\right).$$

This differentiation will produce several terms. The only one that involves $\hat{\mathbf{B}}$ is the one that comes from evaluating $v^2\kappa(d\hat{\mathbf{N}}/dt) = v^3\kappa(d\hat{\mathbf{N}}/ds) = v^3\kappa(\tau\hat{\mathbf{B}} - \kappa\hat{\mathbf{T}})$. Therefore,

$$\frac{d\mathbf{a}}{dt} = \lambda\hat{\mathbf{T}} + \mu\hat{\mathbf{N}} + v^3\kappa\tau\hat{\mathbf{B}},$$

for certain scalars λ and μ. Since $\mathbf{v}\times\mathbf{a} = v^3\kappa\hat{\mathbf{B}}$, it follows that

$$(\mathbf{v}\times\mathbf{a}) \bullet \frac{d\mathbf{a}}{dt} = (v^3\kappa)^2\tau = |\mathbf{v}\times\mathbf{a}|^2\tau.$$

Hence

$$\tau = \frac{(\mathbf{v}\times\mathbf{a}) \bullet (d\mathbf{a}/dt)}{|\mathbf{v}\times\mathbf{a}|^2}.$$

Example 1 Find the curvature, the torsion, and the Frenet frame at a general point on the curve

$$\mathbf{r} = (t + \cos t)\mathbf{i} + (t - \cos t)\mathbf{j} + \sqrt{2}\sin t\mathbf{k}.$$

Describe this curve.

Solution We calculate the various quantities using the recipe given above. First the preliminaries:

$$\begin{aligned}
\mathbf{v} &= (1 - \sin t)\mathbf{i} + (1 + \sin t)\mathbf{j} + \sqrt{2}\cos t\mathbf{k} \\
\mathbf{a} &= -\cos t\mathbf{i} + \cos t\mathbf{j} - \sqrt{2}\sin t\mathbf{k} \\
\frac{d\mathbf{a}}{dt} &= \sin t\mathbf{i} - \sin t\mathbf{j} - \sqrt{2}\cos t\mathbf{k} \\
\mathbf{v}\times\mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - \sin t & 1 + \sin t & \sqrt{2}\cos t \\ -\cos t & \cos t & -\sqrt{2}\sin t \end{vmatrix} \\
&= -\sqrt{2}(1 + \sin t)\mathbf{i} - \sqrt{2}(1 - \sin t)\mathbf{j} + 2\cos t\mathbf{k} \\
(\mathbf{v}\times\mathbf{a}) \bullet \frac{d\mathbf{a}}{dt} &= -\sqrt{2}\sin t(1 + \sin t) + \sqrt{2}\sin t(1 - \sin t) - 2\sqrt{2}\cos^2 t \\
&= -2\sqrt{2} \\
v = |\mathbf{v}| &= \sqrt{2 + 2\sin^2 t + 2\cos^2 t} = 2 \\
|\mathbf{v}\times\mathbf{a}| &= \sqrt{2(2 + 2\sin^2 t) + 4\cos^2 t} = \sqrt{8} = 2\sqrt{2}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{2\sqrt{2}}{8} = \frac{1}{2\sqrt{2}} \\
\tau &= \frac{(\mathbf{v}\times\mathbf{a}) \bullet (d\mathbf{a}/dt)}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{-2\sqrt{2}}{(2\sqrt{2})^2} = -\frac{1}{2\sqrt{2}} \\
\hat{\mathbf{T}} &= \frac{\mathbf{v}}{v} = \frac{1 - \sin t}{2}\mathbf{i} + \frac{1 + \sin t}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\cos t\mathbf{k} \\
\hat{\mathbf{B}} &= \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = -\frac{1 + \sin t}{2}\mathbf{i} - \frac{1 - \sin t}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\cos t\mathbf{k} \\
\hat{\mathbf{N}} &= \hat{\mathbf{B}}\times\hat{\mathbf{T}} = -\frac{1}{\sqrt{2}}\cos t\mathbf{i} + \frac{1}{\sqrt{2}}\cos t\mathbf{j} - \sin t\mathbf{k}.
\end{aligned}$$

Since the curvature and torsion are both constant (they are therefore constant when expressed in terms of any parametrization), the curve must be a circular helix by Theorem 3. It is left-handed, since $\tau < 0$. By Example 3 in Section 11.4, it is congruent to the helix

$$\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k},$$

provided that $a/(a^2 + b^2) = 1/(2\sqrt{2}) = -b/(a^2 + b^2)$. Solving these equations gives $a = \sqrt{2}$ and $b = -\sqrt{2}$, so the helix is wound on a cylinder of radius $\sqrt{2}$. The axis of this cylinder is the line $x = y$, $z = 0$, as can be seen by inspecting the components of $\mathbf{r}(t)$.

■

Example 2 **(Curvature of the graph of a function of one variable)** Find the curvature of the plane curve with equation $y = f(x)$ at an arbitrary point $(x, f(x))$ on the curve.

Solution The graph can be parametrized: $\mathbf{r} = x\mathbf{i} + f(x)\mathbf{j}$. Thus,

$$\begin{aligned} \mathbf{v} &= \mathbf{i} + f'(x)\mathbf{j}, \\ \mathbf{a} &= f''(x)\mathbf{j}, \\ \mathbf{v}\times\mathbf{a} &= f''(x)\mathbf{k}. \end{aligned}$$

Therefore the curvature is

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}.$$

■

Tangential and Normal Acceleration

In the formula obtained earlier for the acceleration in terms of the unit tangent and normal,

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}},$$

the term $(dv/dt)\hat{\mathbf{T}}$ is called the **tangential** acceleration, and the term $v^2\kappa\hat{\mathbf{N}}$ is called the **normal** or **centripetal** acceleration. This latter component is directed toward the centre of curvature and its magnitude is $v^2\kappa = v^2/\rho$. Highway and railway designers attempt to bank curves in such a way that the resultant of the corresponding "centrifugal force," $-m(v^2/\rho)\hat{\mathbf{N}}$, and the weight, $-mg\mathbf{k}$, of the vehicle will be normal to the surface at a desired speed.

Example 3 **Banking a curve.** A level, curved road lies along the curve $y = x^2$ in the horizontal xy-plane. Find, as a function of x, the angle at which the road should be banked (i.e., the angle between the vertical and the normal to the surface of the road) so that the resultant of the centrifugal and gravitational ($-mg\mathbf{k}$) forces acting on the vehicle travelling at constant speed v_0 along the road is always normal to the surface of the road.

Solution By Example 2 the path of the road, $y = x^2$, has curvature

$$\kappa = \frac{|d^2y/dx^2|}{\left(1 + (dy/dx)^2\right)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}.$$

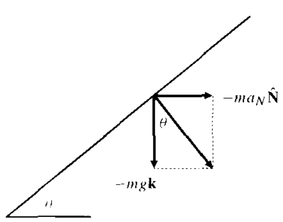

Figure 11.19 Banking a curve on a roadway

The normal component of the acceleration of a vehicle travelling at speed v_0 along the road is

$$a_N = v_0^2\kappa = \frac{2v_0^2}{(1 + 4x^2)^{3/2}}.$$

If the road is banked at angle θ (see Figure 11.19), then the resultant of the centrifugal force $-ma_N\hat{\mathbf{N}}$ and the gravitational force $-mg\mathbf{k}$ is normal to the roadway provided

$$\tan\theta = \frac{ma_N}{mg}, \qquad \text{that is,} \qquad \theta = \tan^{-1}\frac{2v_0^2}{g(1 + 4x^2)^{3/2}}.$$

■

Remark The definition of centripetal acceleration given above is consistent with the one that arose in the discussion of rotating frames in Section 11.2. If $\mathbf{r}(t)$ is the position of a moving particle at time t, we can regard the motion at any instant as being a rotation about the centre of curvature, so that the angular velocity must be $\boldsymbol{\Omega} = \Omega\hat{\mathbf{B}}$. The linear velocity is $\mathbf{v} = \boldsymbol{\Omega}\times(\mathbf{r} - \mathbf{r}_c) = v\hat{\mathbf{T}}$, so the speed is $v = \Omega\rho$, and $\boldsymbol{\Omega} = (v/\rho)\hat{\mathbf{B}}$. As developed in Section 11.2, the centripetal acceleration is

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\mathbf{r} - \mathbf{r}_c)) = \boldsymbol{\Omega}\times\mathbf{v} = \frac{v^2}{\rho}\hat{\mathbf{B}}\times\hat{\mathbf{T}} = \frac{v^2}{\rho}\hat{\mathbf{N}}.$$

Evolutes

The centre of curvature $\mathbf{r}_c(t)$ of a given curve can itself trace out another curve as t varies. This curve is called the **evolute** of the given curve $\mathbf{r}(t)$.

Example 4 Find the evolute of the exponential spiral

$$\mathbf{r} = ae^{-t}\cos t\,\mathbf{i} + ae^{-t}\sin t\,\mathbf{j}.$$

Solution The curve is a plane curve so $\tau = 0$. We will take a shortcut to the curvature and the unit normal without calculating $\mathbf{v}\times\mathbf{a}$. First we calculate

$$\begin{aligned}
\mathbf{v} &= ae^{-t}\Big(-(\cos t + \sin t)\mathbf{i} - (\sin t - \cos t)\mathbf{j}\Big)\\
\frac{ds}{dt} &= v = \sqrt{2}ae^{-t}\\
\hat{\mathbf{T}}(t) &= \frac{1}{\sqrt{2}}\Big(-(\cos t + \sin t)\mathbf{i} - (\sin t - \cos t)\mathbf{j}\Big)\\
\frac{d\hat{\mathbf{T}}}{ds} &= \frac{1}{(ds/dt)}\frac{d\hat{\mathbf{T}}}{dt} = \frac{1}{2ae^{-t}}\Big((\sin t - \cos t)\mathbf{i} - (\cos t + \sin t)\mathbf{j}\Big)\\
\kappa(t) &= \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{1}{\sqrt{2}ae^{-t}}.
\end{aligned}$$

It follows that the radius of curvature is $\rho(t) = \sqrt{2}ae^{-t}$. Since $d\hat{\mathbf{T}}/ds = \kappa\hat{\mathbf{N}}$, we have $\hat{\mathbf{N}} = \rho(d\hat{\mathbf{T}}/ds)$. The centre of curvature is

$$\begin{aligned}
\mathbf{r}_c(t) &= \mathbf{r}(t) + \rho(t)\hat{\mathbf{N}}(t)\\
&= \mathbf{r}(t) + \rho^2\frac{d\hat{\mathbf{T}}}{ds}\\
&= ae^{-t}\Big(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}\Big)\\
&\quad + 2a^2e^{-2t}\frac{1}{2ae^{-t}}\Big((\sin t - \cos t)\mathbf{i} - (\cos t + \sin t)\mathbf{j}\Big)\\
&= ae^{-t}\Big(\sin t\,\mathbf{i} - \cos t\,\mathbf{j}\Big)\\
&= ae^{-t}\Big(\cos(t - \frac{\pi}{2})\mathbf{i} + \sin(t - \frac{\pi}{2})\mathbf{j}\Big).
\end{aligned}$$

Thus, interestingly, the evolute of the exponential spiral is the same exponential spiral rotated 90° clockwise in the plane. (See Figure 11.20(a).)

■

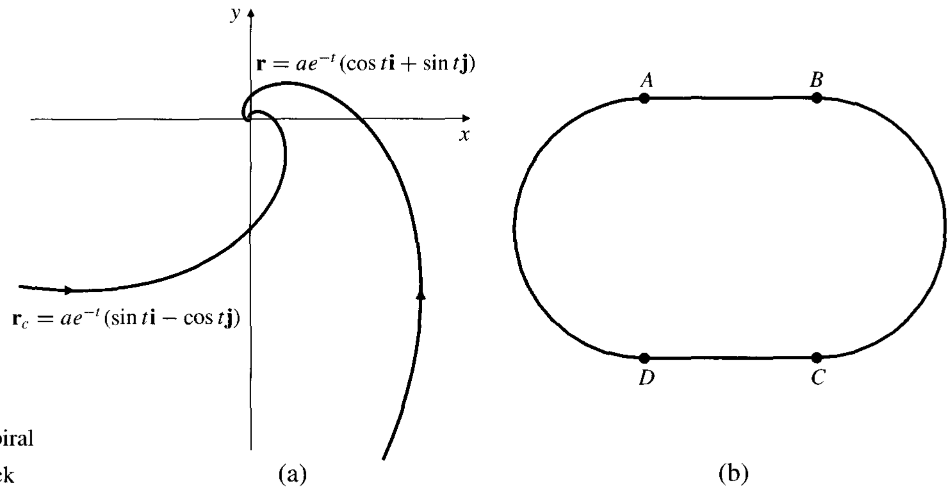

Figure 11.20
(a) The evolute of an exponential spiral is another exponential spiral
(b) The shape of a model train track

An Application to Track (or Road) Design

Model trains frequently come with two kinds of track sections, straight and curved. The curved sections are arcs of a circle of radius R, and the track is intended to be laid out in the shape shown in Figure 11.20(b); AB and CD are straight, and BC and DA are semicircles. The track looks smooth, but is it smooth enough?

The track is held together by friction, and occasionally it can come apart as the train is racing around. It is especially likely to come apart at the points A, B, C, and D. To see why, assume that the train is travelling at constant speed v. Then the tangential acceleration, $(dv/dt)\hat{\mathbf{T}}$, is zero and the total acceleration is just the centripetal acceleration, $\mathbf{a} = (v^2/\rho)\hat{\mathbf{N}}$. Therefore, $|\mathbf{a}| = 0$ along the straight sections, and $|\mathbf{a}| = v^2\kappa = v^2/R$ on the semicircular sections. The acceleration is *discontinuous* at the points A, B, C, and D, and the reactive force exerted by the train on the track is also discontinuous at these points. There is a "shock" or "jolt" as the train enters or leaves a curved part of the track. In order to avoid such stress points, tracks should be designed so that the curvature varies continuously from point to point.

Example 5 Existing track along the negative x-axis and along the ray $y = x - 1$, $x \geq 2$, is to be joined smoothly by track along the transition curve $y = f(x)$, $0 \leq x \leq 2$, where $f(x)$ is a polynomial of degree as small as possible. Find $f(x)$ so that a train moving along the track will not experience discontinuous acceleration at the joins.

Solution The situation is shown in Figure 11.21. The polynomial $f(x)$ must be chosen so that the track is continuous, has continuous slope, and has continuous curvature at $x = 0$ and $x = 2$. Since the curvature of $y = f(x)$ is

$$\kappa = |f''(x)|\Big(1 + (f'(x))^2\Big)^{-3/2},$$

we need only arrange that f, f', and f'' take the same values at $x = 0$ and $x = 2$ that the straight sections do there:

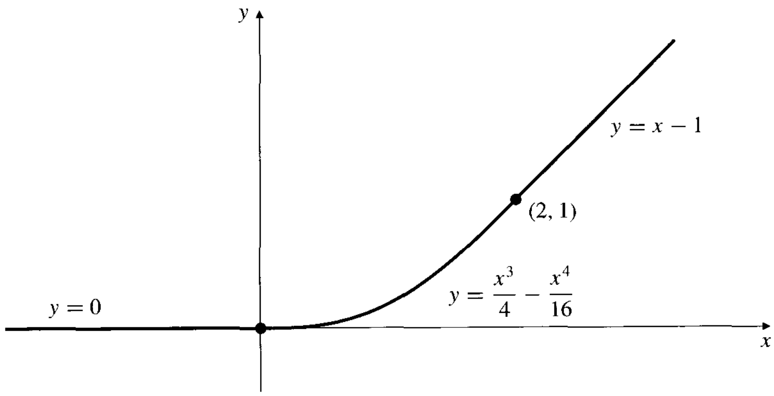

Figure 11.21 Joining two straight tracks with a curved track

$$f(0)=0, \qquad f'(0)=0, \qquad f''(0)=0,$$
$$f(2)=1, \qquad f'(2)=1, \qquad f''(2)=0.$$

These six independent conditions suggest we should try a polynomial of degree 5 involving six arbitrary coefficients:

$$\begin{aligned} f(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \\ f'(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 \\ f''(x) &= 2C + 6Dx + 12Ex^2 + 20Fx^3. \end{aligned}$$

The three conditions at $x = 0$ imply that $A = B = C = 0$. Those at $x = 2$ imply that

$$\begin{aligned} 8D + 16E + \ \ 32F &= f(2) \ = 1 \\ 12D + 32E + \ \ 80F &= f'(2) = 1 \\ 12D + 48E + 160F &= f''(2) = 0. \end{aligned}$$

This system has solution $D = 1/4$, $E = -1/16$, and $F = 0$, so we should use $f(x) = (x^3/4) - (x^4/16)$. ■

Remark Road and railroad builders do not usually use polynomial graphs as transition curves. Other kinds of curves called **clothoids** and **lemniscates** are usually used. (See Exercise 7 in the Review Exercises at the end of this chapter.)

Maple Calculations

Calculations of the sort done in this section for fairly simple curves can become quite oppressive for more complicated curves. As usual, Maple can relieve us of much of this burden. The Maple worksheet **curve-3d.mws** available from the website (`www.pearsoned.ca/divisions/text/adams_calc`) automatically calculates the velocity, acceleration, unit tangent, normal, and binormal vectors, and the curvature and torsion of any given 3-space parametric curve as functions of the parameter of the curve by implementing the formulas in this section. The worksheet reads in **vecops.def** discussed in Section 10.7, so make sure you obtain that file as well from the same source.

The tricky part in dealing with vector-valued functions that you want to differentiate is to define them in such a way that their components are functions. For example, suppose you want to construct the function

$$\mathbf{R} = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}.$$

The Maple input

```
> R := t -> [cos(t), sin(t), t];
```

$$R := t- > [\cos(t), \sin(t), t]$$

will not work, because, although it defines a function whose values are vectors, the function itself is not considered to be a vector. Thus, the input `R[2](t)` does not lead to the output $\sin(t)$ as we would expect, but instead returns the whole vector $\mathbf{R}(t)$. Nor does differentiation of **R** to produce the velocity vector yield the correct result.

```
> R[2](t); V := D(R); V(s);
```

$$[\cos(t), \sin(t), t]$$

$$V := D(R)$$

$$D(R)(s)$$

This is not what we want. We must define **R** in such a way that its components are functions rather than just the vector itself being a function. This can be done by defining x, y, and z separately as functions of t (i.e., `x := t -> cos(t)`, and similar assignments for `y` and `z`), and then defining `R := [x,y,z]`, or we can accomplish the whole thing in one step with the input

```
> R := [t -> cos(t), t -> sin(t), t -> t];
```

$$R := [\cos, \sin, t- > t]$$

Now observe,

```
> R[2](t); V := D(R); V(s)
```

$$\sin(t)$$

$$V := [-\sin, \cos, 1]$$

$$[-\sin(s), \cos(s), 1]$$

We can continue to define the acceleration `A := D(V)` and so on, and construct the various quantities for the curve by standard vector operations. For instance, the speed and curvature functions are defined in **curve-3d.mws** by

```
> spd := t -> len(V(t));
> curv := t -> simplify(len(V(t) &x A(t))/(spd(t))^3);
```

The worksheet **curve-3d.mws** also illustrates the use of the procedure **spacecurve** from the plots package to plot the parametric curve.

Exercises 11.5

Find the radius of curvature of the curves in Exercises 1–4 at the points indicated.

1. $y = x^2$ at $x = 0$ and at $x = \sqrt{2}$
2. $y = \cos x$ at $x = 0$ and at $x = \pi/2$
3. $\mathbf{r} = 2t\mathbf{i} + (1/t)\mathbf{j} - 2t\mathbf{k}$ at $(2, 1, -2)$
4. $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ at the point where $t = 1$

Find the Frenet frames $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}\}$ for the curves in Exercises 5–6 at the points indicated.

5. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$ at $(1, 1, 2)$

6. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ at $(1, 1, 1)$

In Exercises 7–8, find the unit tangent, normal, and binormal vectors, and the curvature and torsion at a general point on the given curve.

7. $\mathbf{r} = t\mathbf{i} + \dfrac{t^2}{2}\mathbf{j} + \dfrac{t^3}{3}\mathbf{k}$ **8.** $\mathbf{r} = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k})$

9. Find the curvature and torsion of the parametric curve

$$x = 2 + \sqrt{2}\cos t, \quad y = 1 - \sin t, \quad z = 3 + \sin t$$

at an arbitrary point t. What is the curve?

10. A particle moves along the plane curve $y = \sin x$ in the direction of increasing x with constant horizontal speed $dx/dt = k$. Find the tangential and normal components of the acceleration of the particle when it is at position x.

11. Find the unit tangent, normal and binormal, and the curvature and torsion for the curve

$$\mathbf{r} = \sin t\cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j} + \cos t\,\mathbf{k}$$

at the points (i) $t = 0$ and (ii) $t = \pi/4$.

12. A particle moves on an elliptical path in the xy-plane so that its position at time t is $\mathbf{r} = a\cos t\mathbf{i} + b\sin t\mathbf{j}$. Find the tangential and normal components of its acceleration at time t. At what points is the tangential acceleration zero?

13. Find the maximum and minimum values for the curvature of the ellipse $x = a\cos t$, $y = b\sin t$, where $a > b > 0$.

14. A bead of mass m slides without friction down a wire bent in the shape of the curve $y = x^2$, under the influence of the gravitational force $-mg\mathbf{j}$. The speed of the bead is v as it passes through the point $(1, 1)$. Find, at that instant, the magnitude of the normal acceleration of the bead and the rate of change of its speed.

15. Find the curvature of the plane curve $y = e^x$ at x. Find the equation of the evolute of this curve.

16. Show that the curvature of the plane polar graph $r = f(\theta)$ at a general point θ is

$$\kappa(\theta) = \frac{\left|2\big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2 - f(\theta)f''(\theta)\right|}{\left[\big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2\right]^{3/2}}.$$

17. Find the curvature of the cardioid $r = a(1 - \cos\theta)$.

*** 18.** Find the curve $\mathbf{r} = \mathbf{r}(t)$ for which $\kappa(t) = 1$ and $\tau(t) = 1$ for all t, and $\mathbf{r}(0) = \hat{\mathbf{T}}(0) = \mathbf{i}$, $\hat{\mathbf{N}}(0) = \mathbf{j}$, and $\hat{\mathbf{B}}(0) = \mathbf{k}$.

19. Suppose the curve $\mathbf{r} = \mathbf{r}(t)$ satisfies $\dfrac{d\mathbf{r}}{dt} = \mathbf{c}\times\mathbf{r}(t)$, where $\mathbf{c}$ is a constant vector. Use curvature and torsion to show that the curve is the circle in which the plane through $\mathbf{r}(0)$ normal to $\mathbf{c}$ intersects a sphere with radius $|\mathbf{r}(0)|$ centred at the origin.

20. Find the evolute of the circular helix $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\mathbf{k}$.

21. Find the evolute of the parabola $y = x^2$.

22. Find the evolute of the ellipse $x = 2\cos t$, $y = \sin t$.

23. Find the polynomial $f(x)$ of lowest degree so that track along $y = f(x)$ from $x = -1$ to $x = 1$ joins with existing straight tracks $y = -1, x \le -1$ and $y = 1, x \ge 1$ sufficiently smoothly that a train moving at constant speed will not experience discontinuous acceleration at the joins.

*** 24.** Help out model train manufacturers. Design a track segment $y = f(x)$, $-1 \le x \le 0$, to provide a jolt-free link between a straight track section $y = 1, x \le -1$, and a semicircular arc section $x^2 + y^2 = 1, x \ge 0$.

*** 25.** If the position $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$ of a moving particle satisfy $\mathbf{a}(t) = \lambda(t)\mathbf{r}(t) + \mu(t)\mathbf{v}(t)$, where $\lambda(t)$ and $\mu(t)$ are scalar functions of time t, and if $\mathbf{v}\times\mathbf{a} \ne \mathbf{0}$, show that the path of the particle lies in a plane.

Use Maple (and, in particular, the procedures defined in the files **curve-3d.mws** and **vecops.def**) in Exercises 26–31.

In Exercises 26–29 determine the curvature and torsion functions for the given curves. Try to determine where the curvature or torsion is maximum or minimum. Use the `spacecurve` routine in the **plots** package to plot the curve.

26. $\mathbf{r}(t) = \cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + \cos(t)\mathbf{k}$. Why should you not be surprised at the value of the torsion? Describe the curve.

27. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + t\mathbf{k}$. Are the curvature and torsion continuous for all t?

28. $\mathbf{r}(t) = \cos(t)\cos(2t)\mathbf{i} + \cos(t)\sin(2t)\mathbf{j} + \sin(t)\mathbf{k}$. Show that the curve lies on the sphere $x^2 + y^2 + z^2 = 1$. What is the minimum value of its curvature?

29. $\mathbf{r}(t) = (t + \cos t)\mathbf{i} + (t + \sin t)\mathbf{j} + (1 + t - \cos t)\mathbf{k}$.

In Exercises 30–31, add new definitions to **curve-3d.mws** to calculate the required functions.

30. A function `evolute(t)` giving the position vector of the centre of curvature of the curve at position `R(t)`.

31. A function `tanline(t,u)` whose value at u is the point on the tangent line to the curve $\mathbf{r} = \mathbf{r}(t)$ at distance u from $\mathbf{r}(t)$ in the direction of increasing t.

11.6 Kepler's Laws of Planetary Motion

The German mathematician and astronomer Johannes Kepler (1571–1630) was a student and colleague of Danish astronomer Tycho Brahe (1546–1601). Over a lifetime of observing the positions of planets without the aid of a telescope, Brahe compiled a vast amount of data, which Kepler analyzed. Although Polish astronomer Nicolaus Copernicus (1473–1543) had postulated that the earth and other planets moved around the sun, the religious and philosophical climate in Europe at the end of the sixteenth century still favoured explaining the motion of heavenly bodies in terms of circular orbits around the earth. It was known that planets such as Mars could not move on circular orbits centred at the earth, but models were proposed in which they moved on other circles (epicycles) whose centres moved on circles centred at the earth.

Brahe's observations of Mars were sufficiently detailed that Kepler realized that no simple model based on circles could be made to conform very closely with the actual orbit. He was, however, able to fit a more general ellipse with one focus at the sun, and, based on this success and on Brahe's data on other planets, he formulated the following three laws of planetary motion:

Kepler's Laws

1. The planets move on elliptical orbits with the sun at one focus.
2. The radial line from the sun to a planet sweeps out equal areas in equal times.
3. The squares of the periods of revolution of the planets around the sun are proportional to the cubes of the major axes of their orbits.

Kepler's statement of the third law actually says that the squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun. The mean distance of points on an ellipse from a focus of the ellipse is equal to the semi-major axis. (See Exercise 17 below.) Therefore the two statements are equivalent.

The choice of ellipses was reasonable once it became clear that circles would not work. The properties of the conic sections were well understood, having been developed by the Greek mathematician Apollonius of Perga around 200 BC. Nevertheless, based, as it was, on observations rather than theory, Kepler's formulation of his laws without any causal explanation was a truly remarkable feat. The theoretical underpinnings came later when Newton, with the aid of his newly created calculus, showed that Kepler's laws implied an inverse square gravitational force. (See Review Exercises 14–16 at the end of this chapter.) Newton believed that his universal gravitational law also implied Kepler's laws, but his writings fail to provide a proof that is convincing by today's standards.[1]

Later in this section we will derive Kepler's laws from the gravitational law by an elegant method that exploits vector differentiation to the fullest. First, however, we need to attend to some preliminaries.

[1] There are interesting articles debating the historical significance of Newton's work by Robert Weinstock, Curtis Wilson, and others in *The College Mathematics Journal,* vol. 25, No. 3, 1994.

Ellipses in Polar Coordinates

The polar coordinates $[r, \theta]$ of a point in the plane whose distance r from the origin is ε times its distance $p - r\cos\theta$ from the line $x = p$ (see Figure 11.22) satisfy the equation $r = \varepsilon(p - r\cos\theta)$, or, solving for r,

$$r = \frac{\ell}{1 + \varepsilon\cos\theta},$$

where $\ell = \varepsilon p$. As observed in Sections 8.1 and 8.5, for $0 \le \varepsilon < 1$ this equation represents an ellipse having **eccentricity** ε. (It is a circle if $\varepsilon = 0$.) To see this, let us transform the equation to Cartesian coordinates:

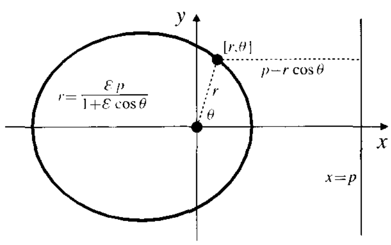

Figure 11.22 An ellipse with focus at the origin, directrix $x = p$, and eccentricity ε

$$x^2 + y^2 = r^2 = \varepsilon^2(p - r\cos\theta)^2 = \varepsilon^2(p - x)^2 = \varepsilon^2(p^2 - 2px + x^2).$$

With some algebraic manipulation, this equation can be juggled into the form

$$\frac{\left(x + \dfrac{\varepsilon\ell}{1-\varepsilon^2}\right)^2}{\left(\dfrac{\ell}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\dfrac{\ell}{\sqrt{1-\varepsilon^2}}\right)^2} = 1,$$

which can be recognized as an ellipse with **centre** at the point $C = (-c, 0)$, where $c = \varepsilon\ell/(1-\varepsilon^2)$, and semi-axes a and b given by

$$a = \frac{\ell}{1-\varepsilon^2} \qquad \textbf{(semi-major axis)},$$
$$b = \frac{\ell}{\sqrt{1-\varepsilon^2}} \qquad \textbf{(semi-minor axis)}.$$

The Cartesian equation of the ellipse shows that the curve is symmetric about the lines $x = -c$ and $y = 0$ and so has a second focus at $F = (-2c, 0)$ and a second directrix with equation $x = -2c - p$. (See Figure 11.23.) The ends of the major axis are $A = (a - c, 0)$ and $A' = (-a - c, 0)$, and the ends of the minor axis are $B = (-c, b)$ and $B' = (-c, -b)$.

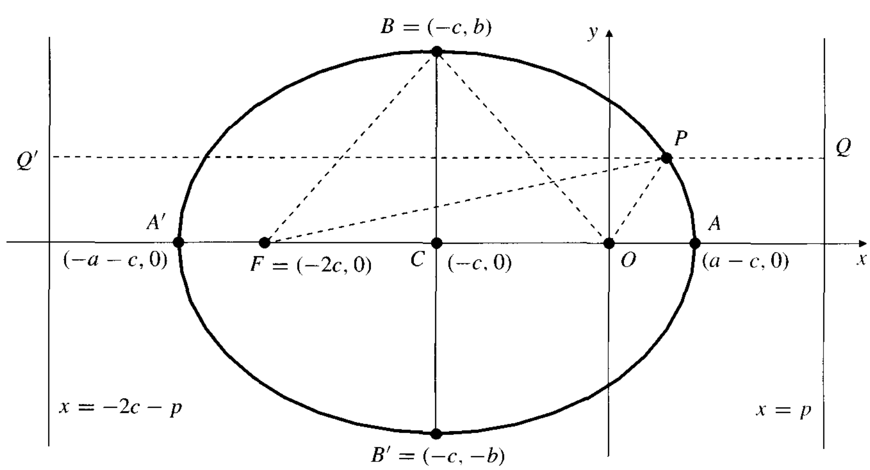

Figure 11.23 The sum of the distances from any point P on the ellipse to the two foci O and F is constant, ε times the distance between the directrices

If P is any point on the ellipse, then the distance OP is ε times the distance PQ from P to the right directrix. Similarly, the distance FP is ε times the distance $Q'P$ from P to the left directrix. Thus, the sum of the focal radii $OP + FP$ is the constant $\varepsilon Q'Q = \varepsilon(2c + 2p)$, regardless of where P is on the ellipse. Letting P be A or B we get for this sum

$$2a = (a - c) + (a + c) = OA + FA = OB + FB = 2\sqrt{b^2 + c^2}.$$

It follows that

$$a^2 = b^2 + c^2, \qquad c = \sqrt{a^2 - b^2} = \frac{\ell\varepsilon}{1 - \varepsilon^2} = \varepsilon a.$$

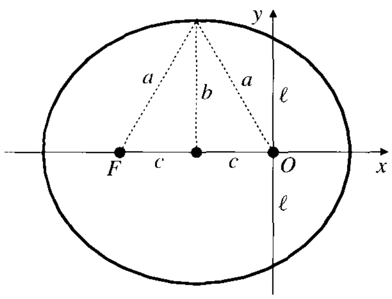

Figure 11.24 Some parameters of an ellipse

The number ℓ is called the **semi-latus rectum** of the ellipse; the latus rectum is the width measured along the line through a focus, perpendicular to the major axis. (See Figure 11.24.)

Remark The polar equation $r = \ell/(1 + \varepsilon\cos\theta)$ represents a *bounded* curve only if $\varepsilon < 1$; in this case we have $\ell/(1 + \varepsilon) \le r \le \ell/(1 - \varepsilon)$ for all directions θ. If $\varepsilon = 1$, the equation represents a parabola, and if $\varepsilon > 1$, a hyperbola. It is possible for objects to travel on parabolic or hyperbolic orbits, but they will approach the sun only once, rather than continue to loop around it. Some comets have hyperbolic orbits.

Polar Components of Velocity and Acceleration

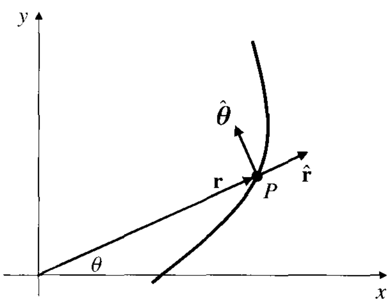

Figure 11.25 Basis vectors in the direction of increasing r and θ

Let $\mathbf{r}(t)$ be the position vector at time t of a particle P moving in the xy-plane. We construct two unit vectors at P, the vector $\hat{\mathbf{r}}$ points in the direction of the position vector $\mathbf{r}$, and the vector $\hat{\boldsymbol{\theta}}$ is rotated 90° counterclockwise from $\hat{\mathbf{r}}$. (See Figure 11.25.) If P has polar coordinates $[r, \theta]$, then $\hat{\mathbf{r}}$ points in the direction of increasing r at P, and $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ. Evidently

$$\hat{\mathbf{r}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

Note that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ do not depend on r but only on θ:

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}}.$$

The pair $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}\}$ forms a reference frame (a basis) at P so that vectors in the plane can be expressed in terms of these two unit vectors. The $\hat{\mathbf{r}}$ component of a vector is called the **radial component**, and the $\hat{\boldsymbol{\theta}}$ component is called the **transverse component**. The frame varies from point to point, so we must remember that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are both functions of t. In terms of this moving frame, the position $\mathbf{r}(t)$ of P can be expressed very simply:

$$\mathbf{r} = r\hat{\mathbf{r}},$$

where $r = r(t) = |\mathbf{r}(t)|$ is the distance from P to the origin at time t.

We are going to differentiate this equation with respect to t in order to express the velocity and acceleration of P in terms of the moving frame. Along the path of motion, $\mathbf{r}$ can be regarded as a function of either θ or t; θ is itself a function of t. To avoid confusion, let us adopt a notation that is used extensively in mechanics and that resembles the notation originally used by Newton in his calculus.

A dot over a quantity denotes the time derivative of that quantity. Two dots denote the second derivative with respect to time. Thus

$$\dot{u} = du/dt \qquad \text{and} \qquad \ddot{u} = d^2u/dt^2.$$

First, let us record the time derivatives of the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. By the Chain Rule, we have

$$\dot{\hat{\mathbf{r}}} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}},$$

$$\dot{\hat{\boldsymbol{\theta}}} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{\mathbf{r}}.$$

Now the velocity of P is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Polar components of velocity

The **radial component of velocity** is $\dot{r}$.
The **transverse component of velocity** is $r\dot{\theta}$.

Since $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are perpendicular unit vectors, the speed of P is given by

$$v = |\mathbf{v}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}.$$

Similarly, the acceleration of P can be expressed in terms of radial and transverse components:

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

Polar components of acceleration

The **radial component of acceleration** is $\ddot{r} - r\dot{\theta}^2$.
The **transverse component of acceleration** is $r\ddot{\theta} + 2\dot{r}\dot{\theta}$.

Central Forces and Kepler's Second Law

Polar coordinates are most appropriate for analyzing motion due to a **central force** that is always directed toward (or away from) a single point, the origin: $\mathbf{F} = \lambda(\mathbf{r})\mathbf{r}$, where the scalar $\lambda(\mathbf{r})$ depends on the position $\mathbf{r}$ of the object. If the velocity and acceleration of the object are $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \dot{\mathbf{v}}$, then Newton's Second Law of Motion ($\mathbf{F} = m\mathbf{a}$) says that $\mathbf{a}$ is parallel to $\mathbf{r}$. Therefore,

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \dot{\mathbf{r}}\times\mathbf{v} + \mathbf{r}\times\dot{\mathbf{v}} = \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

and $\mathbf{r} \times \mathbf{v} = \mathbf{h}$, a constant vector representing the object's angular momentum per unit mass about the origin. This says that $\mathbf{r}$ is always perpendicular to $\mathbf{h}$, so motion due to a central force always takes place in a *plane* through the origin having normal $\mathbf{h}$.

If we choose the z-axis to be in the direction of $\mathbf{h}$ and let $|\mathbf{h}| = h$, then $\mathbf{h} = h\mathbf{k}$, and the path of the object is in the xy-plane. In this case the position and velocity of the object satisfy

$$\mathbf{r} = r\hat{\mathbf{r}} \qquad \text{and} \qquad \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Since $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \mathbf{k}$, we have

$$h\mathbf{k} = \mathbf{r} \times \mathbf{v} = r\dot{r}\hat{\mathbf{r}} \times \hat{\mathbf{r}} + r^2\dot{\theta}\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = r^2\dot{\theta}\mathbf{k}.$$

Hence, for any motion under a central force,

$$r^2\dot{\theta} = h \qquad \text{(a constant for the path of motion).}$$

This formula is equivalent to Kepler's Second Law; if $A(t)$ is the area in the plane of motion bounded by the orbit and radial lines $\theta = \theta_0$ and $\theta = \theta(t)$, then

$$A(t) = \frac{1}{2}\int_{\theta_0}^{\theta(t)} r^2\,d\theta,$$

so that

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2}.$$

Thus, area is being swept out at the constant rate $h/2$, and equal areas are swept out in equal times. Note that this law does not depend on the magnitude or direction of the force on the moving object other than the fact that it is *central*. You can also derive the equation $r^2\dot{\theta} = h$ (constant) directly from the fact that the transverse acceleration is zero:

$$\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0.$$

Example 1 An object moves along the polar curve $r = 1/\theta$ under the influence of a force attracting it toward the origin. If the speed of the object is v_0 at the instant when $\theta = 1$, find the magnitude of the acceleration of the object at any point on its path as a function of its distance r from the origin.

Solution Since the force is central, we know that the transverse acceleration is zero and that $r^2\dot{\theta} = h$ is constant. Differentiating the equation of the path with respect to time and expressing the result in terms of r, we obtain

$$\dot{r} = -\frac{1}{\theta^2}\dot{\theta} = -r^2\frac{h}{r^2} = -h.$$

Hence, the radial component of acceleration is

$$a_r = \ddot{r} - r(\dot{\theta})^2 = 0 - r\frac{h^2}{r^4} = -\frac{h^2}{r^3}.$$

At $\theta = 1$ we have $r = 1$, so $\dot{\theta} = h$. At that instant the square of the speed is

$$v_0^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2 + h^2 = 2h^2.$$

Hence, $h^2 = v_0^2/2$, and, at any point of its path, the magnitude of the acceleration of the object is

$$|a_r| = \frac{v_0^2}{2r^3}.$$

■

Derivation of Kepler's First and Third Laws

The planets and the sun move around their common centre of mass. Since the sun is vastly more massive than the planets, that centre of mass is quite close to the centre of the sun. For example, the joint centre of mass of the sun and the earth lies inside the sun. For the following derivation we will take the sun and a planet as *point masses* and consider the sun to be fixed at the origin. We will specify the directions of the coordinate axes later, when the need arises.

According to Newton's law of gravitation, the force that the sun exerts on a planet of mass m whose position vector is $\mathbf{r}$ is

$$\mathbf{F} = -\frac{km}{r^2}\hat{\mathbf{r}} = -\frac{km}{r^3}\mathbf{r},$$

where k is a positive constant depending on mass of the sun, and $\hat{\mathbf{r}} = \mathbf{r}/r$.

As observed above, the fact that the force on the planet is always directed toward the origin implies that $\mathbf{r}\times\mathbf{v}$ is constant. We choose the direction of the z-axis so that $\mathbf{r}\times\mathbf{v} = h\mathbf{k}$, so the motion will be in the xy-plane and $r^2\dot{\theta} = h$. We have not yet specified the directions of the x- and y-axes but will do so shortly. Using polar coordinates in the xy-plane, we calculate

$$\frac{d\mathbf{v}}{d\theta} = \frac{\dot{\mathbf{v}}}{\dot{\theta}} = \frac{-\dfrac{k}{r^2}\hat{\mathbf{r}}}{\dfrac{h}{r^2}} = -\frac{k}{h}\hat{\mathbf{r}}.$$

Since $d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}}$, we can integrate the differential equation above to find $\mathbf{v}$:

$$\mathbf{v} = -\frac{k}{h}\int \hat{\mathbf{r}}\,d\theta = \frac{k}{h}\hat{\boldsymbol{\theta}} + \mathbf{C},$$

where $\mathbf{C}$ is a vector constant of integration. Therefore, we have shown that

$$|\mathbf{v} - \mathbf{C}| = \frac{k}{h}.$$

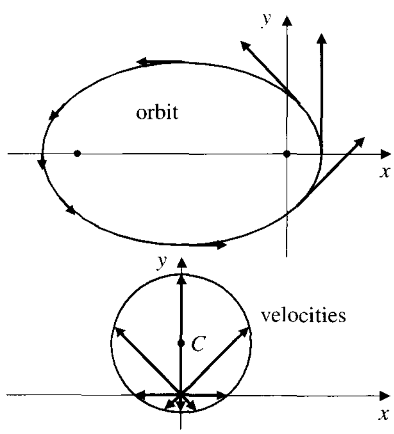

Figure 11.26 The velocity vectors define a circle

This result, known as **Hamilton's Theorem**, says that as a planet moves around its orbit, its velocity vector (when positioned with its tail at the origin) traces out a *circle* with centre point C with position vector $\mathbf{C}$. It is perhaps surprising that there is a circle associated with the orbit of a planet after all. Only it is not the *position* vector that moves on a circle but the *velocity* vector. (See Figure 11.26.)

Recall that so far we have specified only the position of the origin and the direction of the z-axis. Therefore, the xy-plane is determined but not the directions of the x-axis or the y-axis. Let us choose these axes in the xy-plane so that $\mathbf{C}$ is in the direction of the y-axis; say $\mathbf{C} = (\varepsilon k/h)\mathbf{j}$, where ε is a positive constant. We therefore have

$$\mathbf{v} = \frac{k}{h}(\hat{\boldsymbol{\theta}} + \varepsilon\mathbf{j}).$$

The position of the x-axis is now determined by the fact that the three vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular and form a right-handed basis. We calculate $\mathbf{r}\times\mathbf{v}$ again. Remember that $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$, and also $\mathbf{r} = r\hat{\mathbf{r}}$:

$$\begin{aligned} h\mathbf{k} = \mathbf{r}\times\mathbf{v} &= \frac{k}{h}(r\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} + r\varepsilon\cos\theta\mathbf{i}\times\mathbf{j} + r\varepsilon\sin\theta\mathbf{j}\times\mathbf{j}) \\ &= \frac{k}{h}r(1+\varepsilon\cos\theta)\mathbf{k}. \end{aligned}$$

Thus $h = \dfrac{kr}{h}(1+\varepsilon\cos\theta)$, or, solving for r,

$$r = \frac{h^2/k}{1+\varepsilon\cos\theta}.$$

This is the polar equation of the orbit. If $\varepsilon < 1$, it is an ellipse with one focus at the origin (the sun) and with parameters given by

Semi-latus rectum:	$\ell = \dfrac{h^2}{k}$
Semi-major axis:	$a = \dfrac{h^2}{k(1-\varepsilon^2)} = \dfrac{\ell}{1-\varepsilon^2}$
Semi-minor axis:	$b = \dfrac{h^2}{k\sqrt{1-\varepsilon^2}} = \dfrac{\ell}{\sqrt{1-\varepsilon^2}}$
Semi-focal separation:	$c = \sqrt{a^2-b^2} = \dfrac{\varepsilon\ell}{1-\varepsilon^2}$.

We have deduced Kepler's First Law! The choices we made for the coordinate axes result in **perihelion** (the point on the orbit that is closest to the sun) being on the positive x-axis ($\theta = 0$).

Example 2 A planet's orbit has eccentricity ε (where $0 < \varepsilon < 1$) and its speed at perihelion is v_P. Find its speed v_A at **aphelion** (the point on its orbit farthest from the sun).

Solution At perihelion and aphelion the planet's radial velocity $\dot{r}$ is zero (since r is minimum or maximum), so the velocity is entirely transverse. Thus $v_P = r_P\dot{\theta}_P$ and $v_A = r_A\dot{\theta}_A$. Since $r^2\dot{\theta} = h$ has the same value at all points of the orbit, we have

$$r_P v_P = r_P^2 \dot{\theta}_P = h = r_A^2 \dot{\theta}_A = r_A v_A.$$

The planet's orbit has equation

$$r = \frac{\ell}{1 + \varepsilon \cos\theta},$$

so perihelion corresponds to $\theta = 0$ and aphelion to $\theta = \pi$:

$$r_P = \frac{\ell}{1+\varepsilon} \qquad \text{and} \qquad r_A = \frac{\ell}{1-\varepsilon}.$$

Therefore, $v_A = \dfrac{r_P}{r_A} v_P = \dfrac{1-\varepsilon}{1+\varepsilon} v_P.$

We can obtain Kepler's Third Law from the other two as follows. Since the radial line from the sun to a planet sweeps out area at a constant rate $h/2$, the total area A enclosed by the orbit is $A = (h/2)T$, where T is the period of revolution. The area of an ellipse with semi-axes a and b is $A = \pi ab$. Since $b^2 = \ell a = h^2 a/k$, we have

$$T^2 = \frac{4}{h^2} A^2 = \frac{4}{h^2} \pi^2 a^2 b^2 = \frac{4\pi^2}{k} a^3.$$

Note how the final expression for T^2 does not depend on h, which is a constant for the orbit of any one planet, but varies from planet to planet. The constant $4\pi^2/k$ does not depend on the particular planet. (k depends on the mass of the sun and a universal gravitational constant.) Thus,

$$T^2 = \frac{4\pi^2}{k} a^3$$

says that the square of the period of a planet is proportional to the cube of the length, $2a$, of the major axis of its orbit, the proportionality extending over all the planets. This is Kepler's Third Law. Modern astronomical data show that T^2/a^3 varies by only about three-tenths of one percent over the nine known planets.

Conservation of Energy

Solving the second-order differential equation of motion $\mathbf{F} = m\ddot{\mathbf{r}}$ to find the orbit of a planet requires two integrations. In the above derivation we exploited properties of the cross product to make these integrations easy. More traditional derivations of Kepler's laws usually begin with separating the radial and transverse components in the equation of motion:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{r^2}, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

As observed earlier, the second equation above implies that $r^2\dot{\theta} = h =$ constant, which is Kepler's Second Law. This can be used to eliminate θ from the first equation to give

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}.$$

Therefore,

$$\frac{d}{dt}\left(\frac{\dot{r}^2}{2}+\frac{h^2}{2r^2}\right)=\dot{r}\left(\ddot{r}-\frac{h^2}{r^3}\right)=-\frac{k}{r^2}\dot{r}.$$

If we integrate this equation, we obtain

$$\frac{1}{2}\left(\dot{r}^2+\frac{h^2}{r^2}\right)-\frac{k}{r}=E.$$

This is a **conservation of energy** law. The first term on the left is $v^2/2$, the kinetic energy (per unit mass) of the planet. The term $-k/r$ is the potential energy per unit mass. It is difficult to integrate this equation and to find r as a function of t. In any event, we really want r as a function of θ so that we can recognize that we have an ellipse. A way to obtain this is suggested in Exercise 18 below.

Remark The procedure used above to demonstrate Kepler's laws in fact shows that if any object moves under the influence of a force that attracts it toward the origin (or repels it away from the origin) and has magnitude proportional to the reciprocal of the square of distance from the origin, then the object must move in a plane orbit whose shape is a conic section. If the total energy E defined above is negative, then the orbit is *bounded* and must therefore be an ellipse. If $E = 0$, the orbit is a parabola. If $E > 0$, then the orbit is a hyperbola. Hyperbolic orbits are typical for repulsive forces but may also occur for attractions if the object has high enough velocity (exceeding the *escape velocity*). See Exercise 22 below for an example.

Exercises 11.6

1. **(Polar ellipses)** Fill in the details of the calculation suggested in the text to transform the polar equation of an ellipse, $r = \ell/(1+\varepsilon\cos\theta)$, where $0 < \varepsilon < 1$, to Cartesian coordinates in a form showing the centre and semi-axes explicitly.

Polar components of velocity and acceleration

2. A particle moves on the circle with polar equation $r = k$, $(k > 0)$. What are the radial and transverse components of its velocity and acceleration? Show that the transverse component of the acceleration is equal to the rate of change of the speed of the particle.

3. Find the radial and transverse components of velocity and acceleration of a particle moving at unit speed along the exponential spiral $r = e^\theta$. Express your answers in terms of the angle θ.

4. If a particle moves along the polar curve $r = \theta$ under the influence of a central force attracting it to the origin, find the magnitude of the acceleration as a function of r and the speed of the particle.

5. An object moves along the polar curve $r = \theta^{-2}$ under the influence of a force attracting it toward the origin. If the speed of the object is v_0 at the instant when $\theta = 1$, find the magnitude of the acceleration of the object at any point on its path as a function of its distance r from the origin.

Deductions from Kepler's laws

6. The mean distance from the earth to the sun is approximately 150 million km. Halley's Comet approaches perihelion (comes closest to the sun) in its elliptical orbit approximately every 76 years. Estimate the major axis of the orbit of Halley's Comet.

7. The mean distance from the moon to the earth is about 385,000 km, and its period of revolution around the earth is about 27 days (the sidereal month). At approximately what distance from the centre of the earth, and in what plane, should a communications satellite be inserted into circular orbit if it must remain directly above the same position on the earth at all times?

8. An asteroid is in a circular orbit around the sun. If its period of revolution is T, find the radius of its orbit.

* **9.** If the asteroid in Exercise 8 is instantaneously stopped in its orbit, it will fall toward the sun. How long will it take to get there? *Hint:* you can do this question easily if instead you

regard the asteroid as *almost* stopped, so that it goes into a highly eccentric elliptical orbit whose major axis is a bit greater than the radius of the original circular orbit.

10. Find the eccentricity of an asteroid's orbit if the asteroid's speed at perihelion is twice its speed at aphelion.
perihelion and aphelion and the eccentricity of its orbit.

∗ **13.** As a result of a collision, an asteroid originally in a circular orbit about the sun suddenly has its velocity cut in half, so that it falls into an elliptical orbit with maximum distance from the sun equal to the radius of the original circular orbit. Find the eccentricity of its new orbit.

14. If the speeds of a planet at perihelion and aphelion are v_P and v_A, respectively, what is its speed when it is at the ends of the minor axis of its orbit?

15. What fraction of its "year" (i.e., the period of its orbit) does a planet spend traversing the half of its orbit that is closest to the sun? Give your answer in terms of the eccentricity ε of the planet's orbit.

∗ **16.** Suppose that a planet is travelling at speed v_0 at an instant when it is at distance r_0 from the sun. Show that the period of the planet's orbit is

$$T = \frac{2\pi}{\sqrt{k}}\left(\frac{2}{r_0} - \frac{v_0^2}{k}\right)^{-3/2}.$$

Hint: the quantity $\frac{k}{r} - \frac{1}{2}v^2$ is constant at all points of the orbit, as shown in the discussion of conservation of energy. Find the value of this expression at perihelion in terms of the semi-major axis, a.

∗ **17.** The sum of the distances from a point P on an ellipse $\mathcal{E}$ to the foci of $\mathcal{E}$ is the constant $2a$, the length of the major axis of the ellipse. Use this fact in a *geometric* argument to show that the mean distance from points P to one focus of $\mathcal{E}$ is a. That is, show that

$$\frac{1}{c(\mathcal{E})}\int_{\mathcal{E}} r\,ds = a,$$

where $c(\mathcal{E})$ is the circumference of $\mathcal{E}$ and r is the distance from a point on $\mathcal{E}$ to one focus.

∗ **18.** **(A direct approach to Kepler's First Law)** The result of eliminating θ between the equations for the radial and transverse components of acceleration for a planet is

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}.$$

Show that the change of dependent and independent variables:

$$r(t) = \frac{1}{u(\theta)}, \qquad \theta = \theta(t),$$

transforms this equation to the simpler equation

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}.$$

Show that the solution of this equation is

∗ **19.** (What if gravitation were an inverse cube law?) Use the technique of Exercise 18 to find the trajectory of an object of unit mass attracted to the origin by a force of magnitude $f(r) = k/r^3$. Are there any orbits that do not approach infinity or the origin as $t \to \infty$?

20. Use the conservation of energy formula to show that if $E < 0$ the orbit must be bounded; that is, it cannot get arbitrarily far away from the origin.

∗ **21.** **(Polar hyperbolas)** If $\varepsilon > 1$, then the equation

$$r = \frac{\ell}{1 + \varepsilon\cos\theta}$$

represents a hyperbola rather than an ellipse. Sketch the hyperbola, find its centre and the directions of its asymptotes, and determine its semi-transverse axis, its semi-conjugate axis, and semi-focal separation in terms of ℓ and ε.

∗ **22.** **(Hyperbolic orbits)** A meteor travels from infinity on a hyperbolic orbit passing near the sun. At a very large distance from the sun it has speed v_∞. The asymptotes of its orbit pass at perpendicular distance D from the sun. (See Figure 11.27.) Show that the angle δ through which the meteor's path is deflected by the gravitational attraction of the sun is given by

$$\cot\left(\frac{\delta}{2}\right) = \frac{Dv_\infty^2}{k}.$$

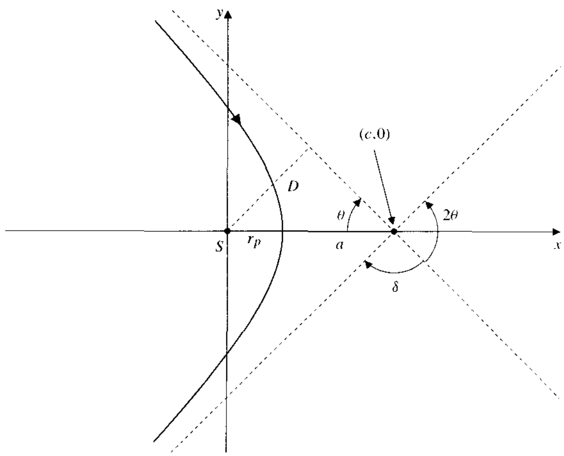

Figure 11.27 Path of a meteor

(*Hint:* you will need the result of Exercise 21.) The same analysis and results hold for electrostatic attraction or repulsion; $f(r) = \pm k/r^2$ in that case also. The constant k depends on the charges of two particles, and r is the distance between them.

Chapter Review

Key Ideas

- **What is a vector function of a real variable, and why does it represent a curve?**
- **State the Product Rule for the derivative of** $\mathbf{u}(t) \bullet \big(\mathbf{v}(t) \times \mathbf{w}(t)\big)$.
- **What do the following terms mean?**
  - ◇ angular velocity
  - ◇ angular momentum
  - ◇ centripetal acceleration
  - ◇ Coriolis acceleration
  - ◇ arc-length parametrization
  - ◇ central force
- **Find the following quantities associated with a parametric curve $\mathcal{C}$ with parametrization $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$.**
  - ◇ the velocity $\mathbf{v}(t)$
  - ◇ the speed $v(t)$
  - ◇ the arc length
  - ◇ the acceleration $\mathbf{a}(t)$
  - ◇ the unit tangent $\hat{\mathbf{T}}(t)$
  - ◇ the unit normal $\hat{\mathbf{N}}(t)$
  - ◇ the curvature $\kappa(t)$
  - ◇ the radius of curvature $\rho(t)$
  - ◇ the osculating plane
  - ◇ the osculating circle
  - ◇ the unit binormal $\hat{\mathbf{B}}(t)$
  - ◇ the torsion $\tau(t)$
  - ◇ the tangential acceleration
  - ◇ the normal acceleration
  - ◇ the evolute
- **State the Frenet–Serret formulas.**
- **State Kepler's laws of planetary motion.**
- **What are the radial and transverse components of velocity and acceleration?**

Review Exercises

1. If $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ represent the position, velocity, and acceleration at time t of a particle moving in 3-space, and if, at every time t, the $\mathbf{a}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{v}$, show that the vector $\mathbf{r}(t) - t\mathbf{v}(t)$ has constant length.

2. Describe the parametric curve
$$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + (2\pi - t)\mathbf{k},$$
$(0 \le t \le 2\pi)$, and find its length.

3. A particle moves along the curve of intersection of the surfaces $y = x^2$ and $z = 2x^3/3$ with constant speed $v = 6$. It is moving in the direction of increasing x. Find its velocity and acceleration when it is at the point $(1, 1, 2/3)$.

4. A particle moves along the curve $y = x^2$ in the xy-plane so that at time t its speed is $v = t$. Find its acceleration at time $t = 3$ if it is at the point $(\sqrt{2}, 2)$ at that time.

5. Find the curvature and torsion at a general point of the curve $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} + e^{-t}\mathbf{k}$.

6. A particle moves on the curve of Exercise 5 so that it is at position $\mathbf{r}(t)$ at time t. Find its normal acceleration and tangential acceleration at any time t. What is its minimum speed?

7. **(A clothoid curve)** The plane curve $\mathcal{C}$ in Figure 11.28 has parametric equations
$$x(s) = \int_0^s \cos\frac{kt^2}{2}\,dt \quad \text{and} \quad y(s) = \int_0^s \sin\frac{kt^2}{2}\,dt.$$
Verify that s is, in fact, arc length along $\mathcal{C}$ measured from $(0, 0)$ and that the curvature of $\mathcal{C}$ is given by $\kappa(s) = ks$. Because the curvature changes linearly with distance along the curve, such curves, called *clothoids*, are useful for joining track sections of different curvatures.

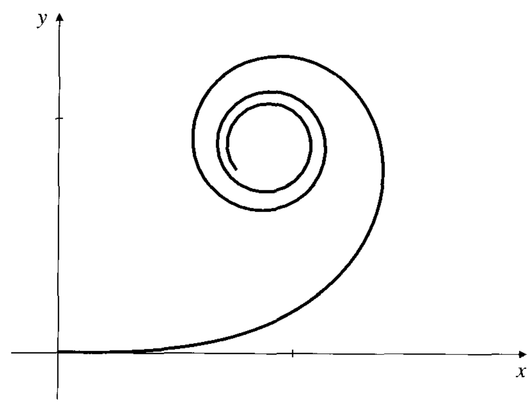

Figure 11.28 A clothoid curve

8. A particle moves along the polar curve $r = e^{-\theta}$ with constant angular speed $\dot{\theta} = k$. Express its velocity and acceleration in terms of radial and transverse components depending only on the distance r from the origin.

Some properties of cycloids

Exercises 9–12 all deal with the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}.$$

Recall that this curve is the path of a point on the circumference of a circle of radius a rolling along the x-axis.

9. Find the arc length $s = s(T)$ of the part of the cycloid from $t = 0$ to $t = T \le 2\pi$.

10. Find the arc-length parametrization $\mathbf{r} = \mathbf{r}(s)$ of the arch $0 \le t \le 2\pi$ of the cycloid, with s measured from the point $(0, 0)$.

11. Find the *evolute* of the cycloid, that is, find parametric equations of the centre of curvature $\mathbf{r} = \mathbf{r}_c(t)$ of the cycloid. Show that the evolute is the same cycloid translated πa units to the right and $2a$ units downward.

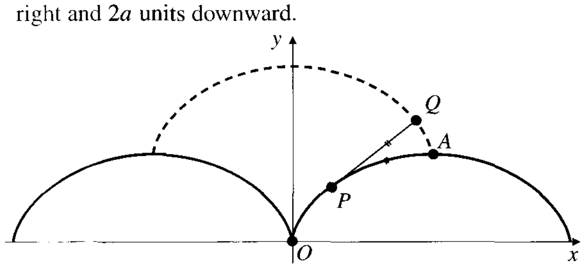

Figure 11.29

12. A string of length $4a$ has one end fixed at the origin and is wound along the arch of the cycloid to the right of the origin. Since that arch has total length $8a$, the free end of the string lies at the highest point A of the arch. Find the path followed by the free end Q of the string as it is unwound from the cycloid and is held taught during the unwinding. (See Figure 11.29.) If the string leaves the cycloid at P, then

$$(\text{arc } OP) + PQ = 4a.$$

The path of Q is called the *involute* of the cycloid. Show that, like the evolute, the involute is also a translate of the original cycloid. In fact, the cycloid is the evolute of its involute.

13. Let P be a point in 3-space with spherical coordinates (ρ, ϕ, θ). Suppose that P is not on the z-axis. Find a triad of mutually perpendicular unit vectors, $\{\hat{\rho}, \hat{\phi}, \hat{\theta}\}$, at P in the directions of increasing ρ, ϕ, and θ, respectively. Is the triad right- or left-handed?

Kepler's laws imply Newton's law of gravitation

In Exercises 14–16, it is *assumed* that a planet of mass m moves in an elliptical orbit $r = \ell/(1 + \varepsilon \cos\theta)$, with focus at the origin (the sun), under the influence of a force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ that depends only on the position of the planet.

14. Assuming Kepler's Second Law, show that $\mathbf{r} \times \mathbf{v} = \mathbf{h}$ is constant and, hence, that $r^2\dot{\theta} = h$ is constant.

15. Use Newton's Second Law of Motion ($\mathbf{F} = m\ddot{\mathbf{r}}$) to show that $\mathbf{r} \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$. Therefore $\mathbf{F}(\mathbf{r})$ is parallel to $\mathbf{r}$: $\mathbf{F}(\mathbf{r}) = -f(\mathbf{r})\,\hat{\mathbf{r}}$, for some scalar-valued function $f(\mathbf{r})$, and the transverse component of $\mathbf{F}(\mathbf{r})$ is zero.

16. By direct calculation of the radial acceleration of the planet, show that $f(\mathbf{r}) = mh^2/(\ell r^2)$, where $r = |\mathbf{r}|$. Thus, $\mathbf{F}$ is an attraction to the origin, proportional to the mass of the planet, and inversely proportional to the square of its distance from the sun.

Challenging Problems

1. Let P be a point on the surface of the earth at 45° north latitude. Use a coordinate system with origin at P and basis vectors $\mathbf{i}$ and $\mathbf{j}$ pointing east and north, respectively, so that $\mathbf{k}$ points vertically upward.

(a) Express the angular velocity $\boldsymbol{\Omega}$ of the earth in terms of the basis vectors at P. What is the magnitude Ω of $\boldsymbol{\Omega}$ in rad/s?

(b) Find the Coriolis acceleration $\mathbf{a}_C = 2\boldsymbol{\Omega} \times \mathbf{v}$ of an object falling vertically with speed v above P.

(c) If the object in (b) drops from rest from a height of 100 m above P, approximately where will it strike the ground? Ignore air resistance but not the Coriolis acceleration. Since the Coriolis acceleration is much smaller than the gravitational acceleration in magnitude, you can use the vertical velocity as a good approximation to the actual velocity of the object at any time during its fall.

* **2. (The spin of a baseball)** When a ball is thrown with spin about an axis that is not parallel to its velocity, it experiences a lateral acceleration due to differences in friction along its sides. This spin acceleration is given by $\mathbf{a}_s = k\mathbf{S} \times \mathbf{v}$, where $\mathbf{v}$ is the velocity of the ball, $\mathbf{S}$ is the angular velocity of its spin, and k is a positive constant depending on the surface of the ball. Suppose that a ball for which $k = 0.001$ is thrown horizontally along the x-axis with an initial speed of 70 ft/s and a spin of 1,000 radians/s about a vertical axis. Its velocity $\mathbf{v}$ must satisfy

$$\begin{cases} \dfrac{d\mathbf{v}}{dt} = (0.001)(1{,}000\mathbf{k}) \times \mathbf{v} - 32\mathbf{k} = \mathbf{k} \times \mathbf{v} - 32\mathbf{k} \\ \mathbf{v}(0) = 70\mathbf{i}, \end{cases}$$

since the acceleration of gravity is 32 ft/s^2.

(a) Show that the components of $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ satisfy

$$\begin{cases} \dfrac{dv_1}{dt} = -v_2 \\ v_1(0) = 70 \end{cases} \qquad \begin{cases} \dfrac{dv_2}{dt} = v_1 \\ v_2(0) = 0 \end{cases} \qquad \begin{cases} \dfrac{dv_3}{dt} = -32 \\ v_3(0) = 0. \end{cases}$$

(b) Solve these equations, and find the position of the ball t s after it is thrown. Assume that it is thrown from the origin at time $t = 0$.

(c) At $t = 1/5$ s, how far, and in what direction, has the ball deviated from the parabolic path it would have followed if it had been thrown without spin?

* **3. (Charged particles moving in magnetic fields)** Magnetic fields exert forces on moving charged particles. If a particle of mass m and charge q is moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, then it experiences a force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, and hence its velocity is governed by the equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}.$$

For this exercise, suppose that the magnetic field is constant and vertical, say $\mathbf{B} = B\mathbf{k}$ (as, e.g., in a cathode-ray tube). If the moving particle has initial velocity $\mathbf{v}_0$, then its velocity at time t is determined by

$$\begin{cases} \dfrac{d\mathbf{v}}{dt} = \omega\mathbf{v} \times \mathbf{k}, & \text{where } \omega = \dfrac{qB}{m} \\ \mathbf{v}(0) = \mathbf{v}_0. \end{cases}$$

(a) Show that $\mathbf{v} \bullet \mathbf{k} = \mathbf{v}_0 \bullet \mathbf{k}$ and $|\mathbf{v}| = |\mathbf{v}_0|$ for all t.

(b) Let $\mathbf{w}(t) = \mathbf{v}(t) - (\mathbf{v}_0 \bullet \mathbf{k})\mathbf{k}$, so that $\mathbf{w}$ is perpendicular to $\mathbf{k}$ for all t. Show that $\mathbf{w}$ satisfies

$$\begin{cases} \dfrac{d^2\mathbf{w}}{dt^2} = -\omega^2\mathbf{w} \\ \mathbf{w}(0) = \mathbf{v}_0 - (\mathbf{v}_0 \bullet \mathbf{k})\mathbf{k} \\ \mathbf{w}'(0) = \omega\mathbf{v}_0 \times \mathbf{k}. \end{cases}$$

(c) Solve the initial-value problem in (b) for $\mathbf{w}(t)$, and hence find $\mathbf{v}(t)$.

(d) Find the position vector $\mathbf{r}(t)$ of the particle at time t if it is at the origin at time $t = 0$. Verify that the path of the particle is, in general, a circular helix. Under what circumstances is the path a straight line? a circle?

* **4. (The tautochrone)** The parametric equations

$$x = a(\theta - \sin\theta) \quad \text{and} \quad y = a(\cos\theta - 1)$$

(for $0 \le \theta \le 2\pi$) describe an arch of the cycloid followed by a point on a circle of radius a rolling along the underside of the x-axis. Suppose the curve is made of wire along which a bead can slide without friction. (See Figure 11.30.) If the bead slides from rest under gravity, starting at a point having parameter value θ_0, show that the time it takes for the bead to fall to the lowest point on the arch (corresponding to $\theta = \pi$) is a *constant*, independent of the starting position θ_0. Thus, two such beads released simultaneously from different positions along the wire will always collide at the lowest point. For this reason, the cycloid is sometimes called the *tautochrone*, from the Greek for "constant time." *Hint:* when the bead has fallen from height $y(\theta_0)$ to height $y(\theta)$, its speed is $v = \sqrt{2g\big(y(\theta_0) - y(\theta)\big)}$. (Why?) The time for the bead to fall to the bottom is

$$T = \int_{\theta=\theta_0}^{\theta=\pi} \frac{1}{v}\,ds,$$

where ds is the arc length element along the cycloid.

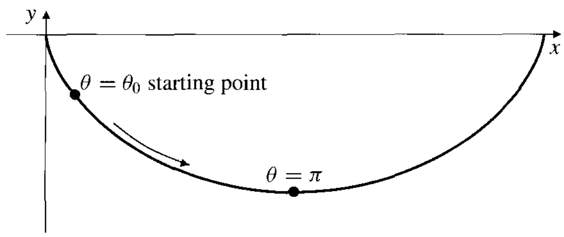

Figure 11.30

* **5. (The Drop of Doom)** An amusement park ride at the West Edmonton Mall in Alberta, Canada, gives thrill seekers a taste of free-fall. It consists of a car moving along a track consisting of straight vertical and horizontal sections joined by a smooth curve. The car drops from the top and falls vertically under gravity for $10 - 2\sqrt{2} \approx 7.2$ m before entering the curved section at B. (See Figure 11.31.) It falls another $2\sqrt{2} \approx 2.8$ m as it whips around the curve and into the horizontal section DE at ground level, where brakes are applied to stop it. (Thus, the total vertical drop from A to D or E is 10 m, a figure, like the others in this problem, chosen for mathematical convenience rather than engineering precision.) For purposes of this problem it is helpful to take the coordinate axes at a 45° angle to the vertical, so that the two straight sections of the track lie along the graph $y = |x|$. The curved section then goes from $(-2, 2)$ to $(2, 2)$ and can be taken to be symmetric about the y-axis. With this coordinate system, the gravitational acceleration is in the direction of $\mathbf{i} - \mathbf{j}$.

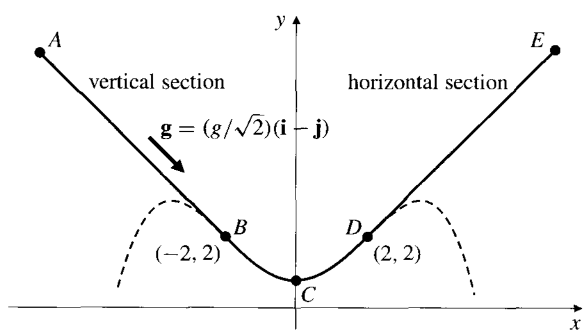

Figure 11.31

(a) Find a fourth-degree polynomial whose graph can be used to link the two straight sections of track without producing discontinuous accelerations for the falling car. (Why is fourth degree adequate?)

(b) Ignoring friction and air resistance, how fast is the car moving when it enters the curve at B? at the midpoint C of the curve? and when it leaves the curve at D?

(c) Find the magnitude of the normal acceleration and of the total acceleration of the car as it passes through C.

* **6.** **(A chase problem)** A fox and a hare are running in the xy-plane. Both are running at the same speed v. The hare is running up the y-axis; at time $t = 0$ it is at the origin. The fox is always running straight toward the hare. At time $t = 0$ the fox is at the point $(a, 0)$, where $a > 0$. Let the fox's position at time t be $(x(t), y(t))$.

(a) Verify that the tangent to the fox's path at time t has slope

$$\frac{dy}{dx} = \frac{y(t) - vt}{x(t)}.$$

(b) Show that the equation of the path of the fox satisfies the equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Hint: differentiate the equation in (a) with respect to t. On the left side note that $(d/dt) = (dx/dt)(d/dx)$.

(c) Solve the equation in (b) by substituting $u(x) = dy/dx$ and separating variables. Note that $y = 0$ and $u = 0$ when $x = a$.

7. Suppose the earth is a perfect sphere of radius a. You set out from the point on the equator whose spherical coordinates are $(\rho, \phi, \theta) = (a, \pi/2, 0)$ and travel on the surface of the earth at constant speed v, always moving toward the northeast (45° east of north).

(a) Will you ever get to the north pole? If so, how long will it take to get there?

(b) Find the functions $\phi(t)$ and $\theta(t)$ that are the angular spherical coordinates of your position at time $t > 0$.

(c) How many times does your path cross the meridian $\theta = 0$?